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POLYOMINOES 2.2

HOW THEY FIT TOGETHER.

FIRST PUBLISHED: 1 MAY 2018
LAST MODIFIED: 19 MAY 2019

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Preface

On 8 September 2017 I set out to discover as much about polyominoes as I could. I thought it would be nice to collect what I found in a few essays, and this collection is the result.

In putting these essays together, I tried to keep the mathematics simple; you will not need to know much graph theory, combinatorics, or other machinery to understand them. At times this means I cover only a very concrete special case of a more general (and arguably beautiful) result, and some proofs in their naive disguise may be a bit clunky; this is the trade-off. I *do* try to give as many references to the bigger theory as possible.

These essays are work in progress. This means I have not yet caught every error or included every reference, and I omitted or neglected (for now) some important topics. If you spot an error or have any suggestions, please let me know [1].

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Currently, I am fleshing out some of the newer sections. But I hope to add more to this; some topics I am considering are *Enumeration*, *Algorithms*, *Compatibility*, and *Tatami Tilings*.

Problems. There are three types of problems:

- Exercises that are interesting ideas or problems that occurred to me as I worked through the topics that I thought is helpful in building intuition and understanding, but not important enough for the main text.

- Questions that occurred to me that I have not solved. Sometimes these are ideas for further investigation on which I did not spend any time; in other cases these are problems I could not solve even after some effort.

- Open questions from the literature that usually has been open for some time; most of these can be considered very difficult.

I plan to mark these in future version, but for now these are all just "Problems".

1.1 *Acknowledgments*

I want to thank Justin Southey for reading a draft of the first essay. He made many suggestions that improved it greatly.

1.2 *Notes*

Numbers in orange like **A000105** refer to integer sequences in the *Online Encyclopedia of Integer Sequences* (Sloane). Clicking on them takes you to the relevant entry.

Notation	Meaning
$\|R\|$	The area of R, the number of cells of R
$\Delta(R)$	Deficiency of R
$\phi(R)$	The flow of R
$\mathcal{W}(R), \mathcal{B}(R)$	The set of white (black) cells of R
$w(S), b(S)$	The number of dominoes with one white (black) cell inside S and the other outside
$G_{\mathcal{T}}(R), G(R)$	The gap number of region R with respect to tileset $\mathcal{T}$
$\rho(T)$	The ranking of a tiling
$\mu(T, U)$	The number of moves from tiling T to U
$\#_{\mathcal{T}}R, \#R$	The number of tilings of a region R by tiles from the $\mathcal{T}$

Table 1: Notation

2

Introduction

2.1 A Quick Introduction to Polyominoes and Tilings

POLYOMINOES[1] ARE SHAPES formed by "gluing" 1×1 squares together, edge-to-edge. Figures 1 to 7 show the polyominoes with five cells or less. The squares that make up a polyomino are called **cells**, and we classify polyominoes according to the number of cells they have. The names for the smaller classes of polyominoes is given in Table 2. We will use the notation $\mathcal{P}$for the set of all polyominoes, and $\mathcal{P}\backslash$ for the set of polyominoes with n cells.

n	Name
1	Monomino
2	Domino
3	Tromino
4	Tetronino
5	Pentomino

Table 2: Names for classes of polyominoes

[1] A polyomino with n cells is also called an *n-omino*.

These simple shapes give rise to a variety of interesting problems. The one which will be the main topic of these essays, is: Which shapes can we build if we put them together so that edges line up?

Problems such as these are called *tiling problems*. A different way to put a tiling problem is: given some shape (we call this a **region**[2]) and a set of other shapes (called the **tiles**), can we cover the region completely with copies of the tiles so that no piece of tile falls outside the region, no tiles overlap? If this is possible, we say the region can be tiled by the set of tiles. Figure 5 shows some examples of tilings.

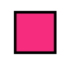


Figure 1: Monomino

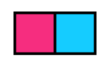


Figure 2: Domino

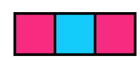


Figure 3: Bar tromino

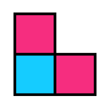


Figure 4: Right tromino

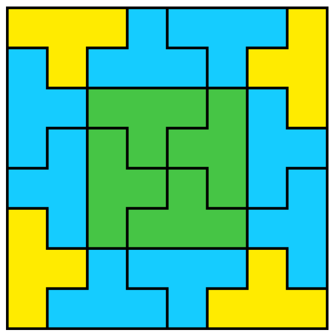
[2] Also called *figure* or *picture*.

Tiling is a big and difficult topic in mathematics, but we can get some insight by limiting our study to polyominoes.

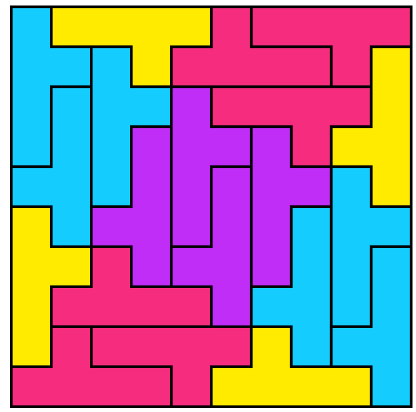
In our case, we will only work with regions that are also composed from squares glued edge-to-edge. Regions are often polyominoes themselves, but we will at times also consider infinite regions, or disconnected regions, or regions with barriers that tiles are not allowed to cross.

The number of cells of a finite region R is denoted by $|R|$. Since each cell is has an area of 1, $|R|$ also equals the area.

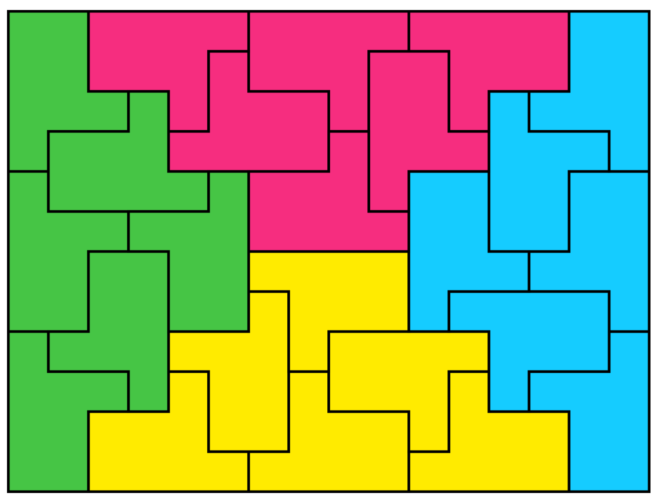
In a region or a polyomino, we call two cells **neighbors** if they share an edge. A cell with four neighbors is called an **interior cell**; otherwise it is called a **border cell**. A cell with only one or two neighbors is called a **corner cell**.



(a) A tiling of a rectangle by the T tetromino.

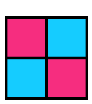


(b) A tiling of a rectangle by Y pentominoes.

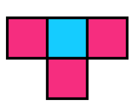


(c) A tiling of a rectangle by a heptomino.

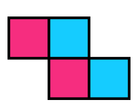
Figure 5: Examples of tilings.



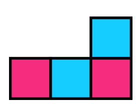
(a) Square



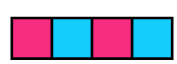
(b) T



(c) Skew

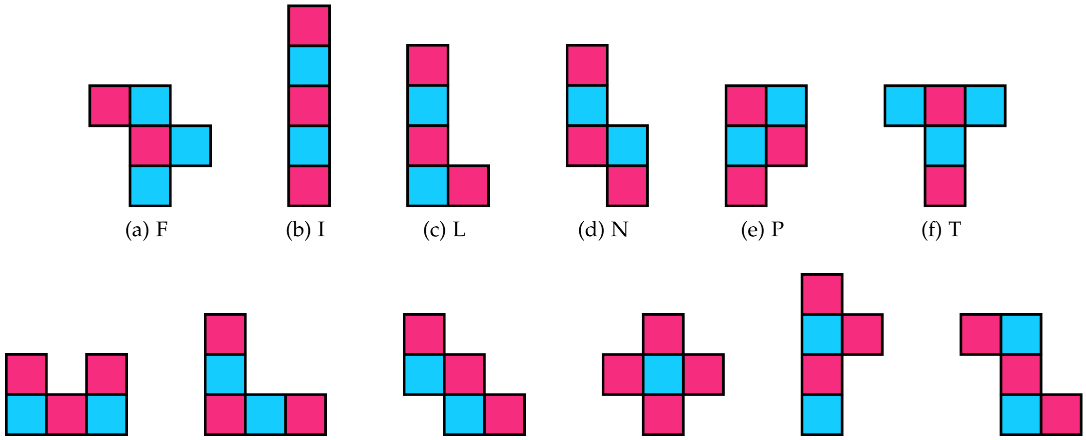


(d) L



(e) Bar

Figure 6: Tetrominoes



(a) F (b) I (c) L (d) N (e) P (f) T

(g) U (h) V (i) W (j) X (k) Y (l) Z

Figure 7: Pentominoes

Two cells are **connected** if we can move from neighbor to neighbor from one to the other. We call a region where every cell is connected to every other cell **connected**, and if it has no holes[3], we call it **simply-connected**.[4] Figure 8 shows examples of polyominoes that are not simply-connected as we use the term here.

To conclude this quick introduction, I state two obvious theorems that we will reference often. I omit their proofs.

Theorem 1 (Area Criterion)**.** *If we have a tile set where all tiles have area n, then we can only tile regions whose area is divisible by n.*

[Referenced on pages 13, 27, 30, 50, 90, 101, 106, 108, 119, 155, 166, 194, 197 and 210]

Example 1. *If a region is tileable by dominoes, then its area is even.*

If S is a subset of the cells of a region R, we say S is a **subregion** of R, (see Figure 9), and if in addition $S \neq R$, S is a **proper subregion** of R. The set of cells in R not in S is denoted $R - S$. A partition of a region R is a collection of subregions S_i of R such that every cell of R is contained in exactly one of S_i . To **partition** a region means to give a partition of the region.

Theorem 2 (Partitions)**.** *If all the subregions in a partition of a region are tileable, then so is the region.*

[Referenced on pages 49, 54, 56, 58, 59, 117 and 118]

Example 2. *There is a region with $2n$ cells not tileable by $\mathcal{P}\backslash$ for all n.*

Consider a cross with area $2n$ and arms of length $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$. If we place a polyomino to cover one arm, it must cover the center of the cross, and another arm. This partitions the cross into three pieces—the covered

[3] Although it is intuitively clear what a hole is, it is a bit harder to give a precise definition, and we will not do this here.

[4] Polyominoes with holes are also called *holey*; polyominoes without holes are also called *simple* (Herzog et al., 2015) or *profane* (Toth et al., 2017, p. 367). Some authors use a slightly different notion of simply-connected, so that the second polyomino in Figure 8 is simply connected.

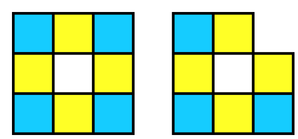


Figure 8: Two examples of polyominoes with holes.

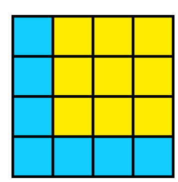


Figure 9: In this example, the 4×4 square is the region, and both the yellow and blue regions are subregions of the square.

arms and the two non-covered arms. Each of the non-covered arms is smaller than n and non-zero, so by Theorem 1 cannot be tiled. And hence, the whole figure cannot be tiled.

2.2 *Symmetry*

Generally, we allow polyominoes to be rotated or reflected any way possible for tiling problems, and therefor we do not distinguish between different orientations or reflections of a polyomino. In this case, we call the polyominoes **free** (Redelmeier, 1981, Section 3). If we distinguish reflections, but not rotations, we call the polyominoes **one-sided** (Golomb, 1996, p. 70). If we distinguish between both reflections and rotations, we call the polyominoes **fixed** (Redelmeier, 1981, Section 3). With this terminology, there are 7 one-sided tetrominoes, and 19 fixed tetrominoes.

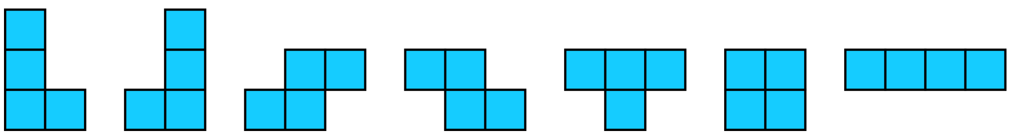


Figure 10: One-sided tetrominoes.

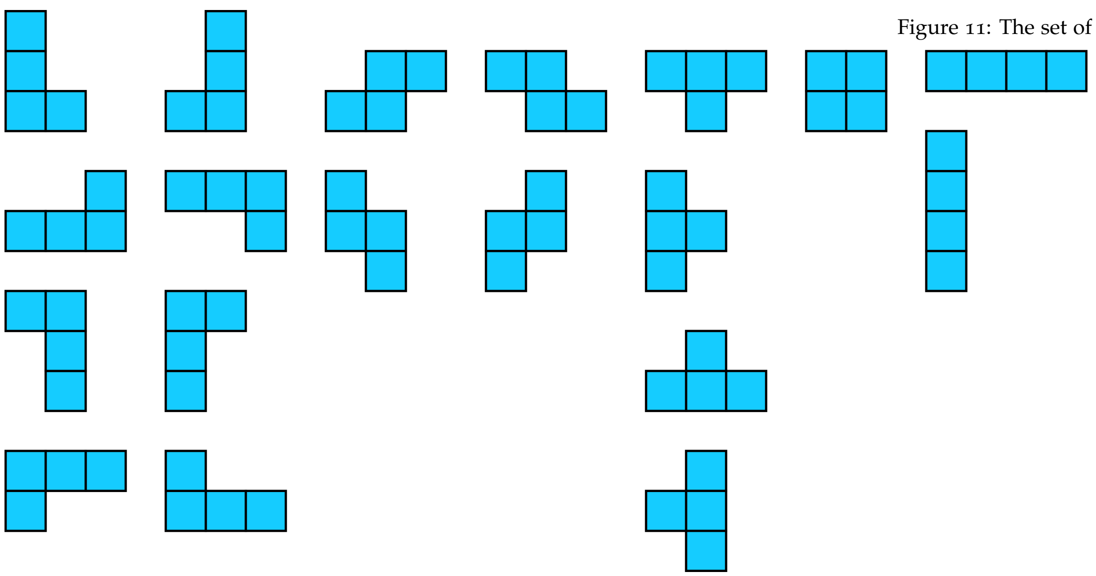


Figure 11: The set of fixed tetrominoes.

If a transformation leaves a free polyomino unaffected, we say the polyomino has the symmetry associated with that transformation. For example, the skew polyomino is not affected by 180° rotations, so it has 180°-rotational symmetry.

The **symmetry index** of a (free) polyomino is the number of fixed polyominoes congruent to it (Redelmeier, 1981, Section 3).

Table 3 shows details of the symmetry types, including an example of each.

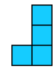
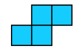
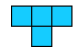
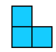
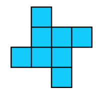
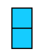
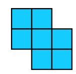
Type	Index	Symmetries	Smallest Example
None	8		
Rot	4	180° rotation	
Axis	4	horizontal or vertical reflection	
Diag	4	diagonal-reflection	
Rot2	2	90° rotation	
Axis2	2	horizontal and vertical reflection or 180° rotation	
Diag2	2	diagonal-reflection or 180° rotation	
All	1	all	

Table 3: The symmetry types of free polyominoes.

Tilings can also be classified by their symmetries in a similar way. A tiling of a region R cannot have more symmetries than R.

Theorem 3. *If a region has a tiling by a polyomino with symmetry type* **All**, *the tiling is unique.*

[Referenced on pages 166 and 201]

Proof. (Adapted from nickgard (2017).) The left-most cell in the top row of the region can only be tiled by the left-most cell of the top row of the tile in any orientation (which are all equivalent because the tile has symmetry type **All**). We can form a new region from the untiled part, which by a similar argument can only be tiled one way. We can repeat this until eventually the complete figure is tiled. Since in each step only one placement is possible, the entire tiling is unique. □

Since squares have symmetry type **All**, it follows that if a region has a tiling by a single square, the tiling is unique.

Problem 1. *Show that for a polyomino* $P \in$ **All** *with perimeter* p,

(1) $|P| \equiv 0$ *or* $1 \pmod 4$

(2) $p(P) \equiv 0 \pmod 4$.

Problem 2. *A* ***Baiocchi figure*** *of a polyomino is a region with symmetry type* **All** *that is tileable by that polyomino*[5]. *A Baiocchi figure of a polyomino*

[5] The idea was suggested by Claudio Baiocchi in January 2008, and appeared in that month in Friedman (2008).

*is **minimal** if there is no smaller Baiocchi figure for that polyomino. Figure 12 shows minimal Baiocchi figures for the tetrominoes, and Figure 13 shows holeless variants.*

(1) *Find minimal Baiocchi figures for the pentominoes.*

(2) *Is there a Baiocchi figure for the U-pentomino without holes? This seems to be an open problem (Sicherman, 2015).*

(3) *Find the minimal holeless Baiochhi figures for other pentominoes.*

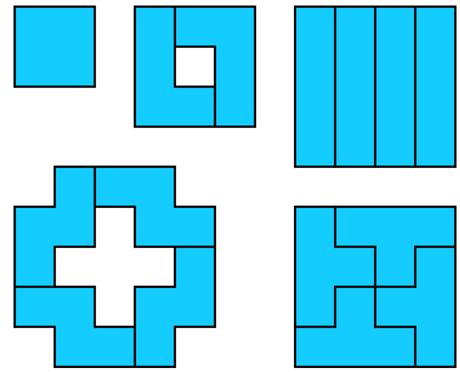


Figure 12: Minimal Baiocchi figures for the tetrominoes (Sicherman, 2015).

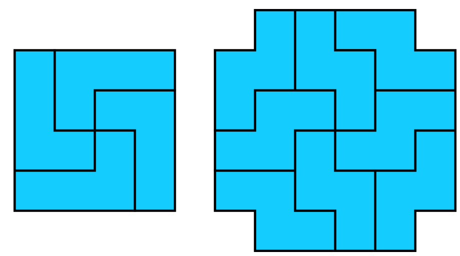


Figure 13: Holeless variants for minimal Baiocchi figures for the L-tetromino and skew tetromino (Sicherman, 2015).

2.3 *The Geometry and Topology of a Polyomino*

(Note: this section needs to be reworked completely; as it stands it has a lot of problems, especially with vague definitions that make the proofs unconvincing. I left this section in, because some of the ideas here are relevant to other sections.)

A **rectilinear polygon**[6] is a polygon with all edges meeting at right angles. A polyomino, therefor, is a rectilinear polygon whose edges are integer lengths. We prove a few simple facts about rectilinear polygons that will be useful to our study of polyominoes.

[6] Also called *orthogonal polygon* or *axis-aligned*.

Theorem 4 (Wikipedia contributors (2017)). *In a simply-connected rectilinear polygon, the number of vertical and horizontal edges are equal.*

[Referenced on pages 16 and 17]

Proof. Each vertical edge is followed by a horizontal edge, and vice versa. □

Theorem 5. *In a polyomino the total horizontal edge length and total vertical edge length are even, and so is the perimeter.*

[Referenced on pages 16, 21 and 37]

Proof. If traveling along the perimeter we go x units left, and x' units right, then $x + x' = 0$ (because the perimeter is a closed finite curve, so we eventually end up where we started). Then the total horizontal distance is $|x| + |x'| = 2|x|$, which is even since x is an integer. The same goes for the vertical length.

Since the perimeter is the sum of the vertical and horizontal length, which are both even, it must be even too. □

Theorem 6 (Csizmadia et al. (2004), Lemma 3.3). *If the lengths of either all horizontal or all vertical edges are odd, then there is an even number of horizontal and vertical edges, and the total number of edges is divisible by four.*

[Referenced on page 37]

Proof. WLG suppose the lengths of all horizontal sides are odd. Since the total length must be even (Theorem 5), there must be an even number of them, and because the number of horizontal and vertical sides are equal (Theorem 4), the total must be divisible by four. □

Problem 3 (Sallows et al. (1991)). *A golygon*[7] *is a rectiliear polygon whose edge lengths forms a sequence* $1, 2, 3, 4, \cdots$; *usually, self-intersections are allowed. An example with 8 sides is shown in Figure 14. Show the number of edges of an golygon must be divisible by 8.*

The golygon in Figure 14 is the only one with 8 sides. It is the only known golygon that can tile the plane. It seems to be an open problem whether any other golygons can tile the plane (Sallows, 1992).

[7] Also called a 90°*serial isogon*.

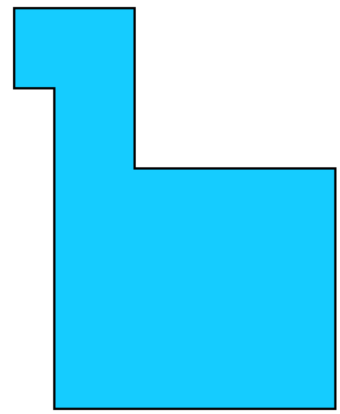


Figure 14: Golygon with 8 sides. This isogon tiles the plane.

Theorem 7. *In a simply-connected rectilinear polygon, the number of edges is even.*

[Referenced on pages 19 and 37]

Proof. Since there is the same number of vertical edges as horizontal edges (Theorem 4), their sum is even. □

A corner with interior angle of 90° is called **convex** and a corner with an interior angle of 270° is called **concave** Bar-Yehuda and Ben-Hanoch (1996).

Theorem 8 (Wikipedia contributors (2017)). *In a finite, simply-connected rectilinear polygon, the number of convex corners is 4 more than the number of concave corners.*

[Referenced on pages 18 and 116]

Proof 1. Suppose we travel anti-clockwise around the perimeter. At concave corners, we need to make a left turn, and at a convex corner, we need to make a right turn. When we return to our starting point, we face the original direction, after having made a net turn of 360 degrees, which means we had to make at least 4 right turns, or traversed 4 convex corners. □

Proof 2. The sum of interior angles of a polygon with k sides is $180^\circ(k-2)$. Suppose we have m convex corners and n convex corners. We then have the equation $90^\circ m + 270^\circ n = 180^\circ(k-2) = 180^\circ(m+n-2)$. Solving this, we obtain $m = n + 4$. □

It follows that any finite rectilinear polygon must have at least 4 convex corners. If the region is not finite, it may not have any corners. And if we allow holes, we can have more concave corners than convex corners.

Problem 4.

(1) *Give examples of regions (not necessarily finite or simply-connected) with:*

(a) *No corners.*

(b) *One convex corner and no concave corners.*

(c) *Two convex corners and no concave corners.*

(d) *Four concave corners and no convex corners.*

(2) *Prove the following regions are impossible. Regions with:*

(a) *Three convex corners and no concave corners.*

(b) *One, two, or three concave corners and no convex corners.*

(3) *Prove that all other combinations are possible.*

Problem 5. *Prove that (up to symmetry) there is a unique rectilinear figure with 6 sides whose side lengths are given.*

Example 3. *Suppose we can place n 2×2-squares anywhere on the lattice. They are allowed to overlap but not coincide. Then they must cover at least $n + 3$ cells.*

Here is why: Any cell can be covered by at most 4 squares. Convex corners can only be covered by one square. There must be at least 4 convex corners (Theorem 8). So all possible cells that can potentially be covered by 4 squares is given by $\frac{4n-4}{4} = n - 1$. So there must be at least $n - 1 + 4 = n + 3$ cells in total.

An edge is called a **mountain** if lies between two convex corners, a **valley** if lies between two concave corners, and a **flat** otherwise.[8]

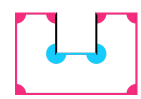


Figure 15: In this region, convex corners are red, and concave corners are blue. The five mountains that lie between pairs of convex corners are marked red, and the valley between the pair of concave corners is blue. The two remaining (black) edges are flats.

[8] In Bar-Yehuda and Ben-Hanoch (1996) the author calls a mountain a *knob* and a valley an *anti-knob*. The term *peak* is used in Beauquier et al. (1995) for a similar concept, so I chose to use the term *mountain*, since it allows the natural extension to *valleys* and *flats*.

Theorem 9 (ccorn (2017) via Wikipedia contributors (2017))**.** *Let M be the number of mountains and V the number of valleys of a finite simply-connected region. Then $M - V = 4$.*

[Referenced on page 19]

Proof. Let n_x be the number of convex corners, and n_y the number of concave corners. Let n_{xx} be the number of convex corners followed by a convex corner, n_{xy} the number of convex corners followed by a concave corner, and so on.

Then $n_x = n_{xx} + n_{xy} = n_{xx} + n_{yx}$ and $n_y = n_{yx} + n_{yy} = n_{xy} + n_{yy}$, which means $n_{xx} = n_x - n_{xy} = n_x - (n_y - n_{yy}) = n_{yy} + (n_x - n_y)$. But by Theorem 8, $n_x - n_y = 4$, so $n_{xx} = n_{yy} + 4$. □

It follows that for finite simply-connected regions, the number of mountains must always be at least 4. Furthermore:

Theorem 10. *If F is the number of flats for a finite simply-connected region, then F is even.*

[Not referenced]

Proof. Since $M - V = 4$ (Theorem 9), it follows that M and V must either both be odd or both be even. So their sum is even. But $M + V + F = k$, and k is even (Theorem 7), therefor F must be even. □

The following theorems, although they seem intuitively true, require a more precise definition of both polyomino and hole. I state them here for completeness; see the reference for details.

Theorem 11 (Herzog et al. (2014), Lemma 1.1)**.** *Every hole is simply-connected.*

[Not referenced]

Theorem 12 (Herzog et al. (2014), Lemma 1.2)**.** *In a simply-connected polyomino, if a vertex is shared between exactly two cells, the cells are neighbors; that is, they cannot be diagonally opposite.*

[Referenced on page 21]

Theorem 13. *For a rectilinear polygon, the number of holes H is related to the number of mountains P and valleys V by the following equation:*

$$H = \frac{V - M}{4} + 1.$$

[Referenced on page 115]

Proof. Suppose we have holes $1, 2, 3, \ldots, H$, and the border of hole i has M_i mountains and V_i valleys, and the outer border of our polygon has M_0 mountains and V_0 valleys. The total mountains and valleys is given by:

$$M = \sum_{i=0}^{H} M_i \text{ and}$$
$$V = \sum_{i=0}^{H} V_i.$$

And so

$$V - M = \sum_{i=0}^{H} (V_i - M_i).$$

If we consider the border of a hole as a rectilinear region on its own, we notice that its mountains correspond to valleys in the original polygon, and its valleys corresponds to mountains in the original polygon. So it has V_i mountains and P_i valleys, and so we know by Theorem 9 that $V_i - M_i = 4$, for $i > 0$. For $i = 0$ we have $M_0 - V_0 = 4$, and so we get the following:

$$\begin{aligned} V - M &= V_0 - M_0 + \sum_{i=1}^{H} (V_i - M_i) \\ &= -4 + \sum_{i=1}^{H} 4 \\ &= -4 + 4H \\ &= 4(H-1), \end{aligned}$$

and so

$$H = \frac{V - M}{4} + 1.$$

□

A **lattice polygon** is a polygon whose vertices fall on a lattice.

Theorem 14 (Pick's Theorem, Niven and Zuckerman (1967), Theorem 4)**.** *The area A of a lattice polygon, is given by $A = b/2 + r - 1$, where b is the number of lattice points on the boundary of the polygon, and r is number of interior vertices.*[9]

[9] This theorem was discovered by Pick (Pick (1899)); Bruckheimer and Arcavi (1995) provides some history and context; more references are given in the further reading section.

[Referenced on pages 21 and 232]

Proof. The details of the following outline is easy to fill in:

(1) The theorem is true for rectangles.

(2) The theorem is true for right triangles.

(3) The theorem is true for triangles.

(4) The theorem is true for polygons.

For details, see the reference. □

Theorem 15 (Williams and Thompson (2008), Theorem 1)**.** *If the number of interior vertices of a simply-connected n-omino is given by r, the perimeter p is given by:*

$$p = 2n - 2r + 2.$$

[Referenced on pages 21 and 33]

Proof. We use induction on n.

The theorem is true for the monomino and domino.

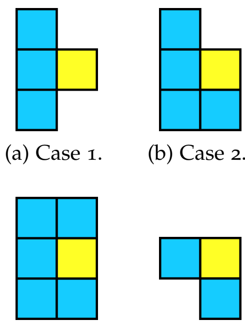
Suppose then it is true for a polyomino with n cells and r interior vertices, so that the perimeter $p = 2n - 2r + 2$.

We can attach another cell to the polyomino to form a new polyomino with $n+1$ cells. Let p' be the perimeter of this polyomino, r' be the number of interior vertices of the new polyomino. We want to show that $p' = 2(n+1) - 2r + 2$.

There are four possibilities (shown in Figure 16).

- Case 1: No new interior vertices are added. The new perimeter is 2 units longer; so $p' = p' + 2 = 2n - 2r + 4 = 2(n+1) - 2r' + 2$.
- Case 2: The perimeter stays the same, but one more interior vertex is added. So $p' = p = 2n - 2r + 2 = 2(n+1) - 2(r+1) + 2 = 2(n+1) - 2r' + 2$.
- Case 3: In this case two new interior vertices are added, and the perimeter is reduced by 2. Thus $p' = p - 2 = 2n - 2r + 2 = 2(n+1) + 2(r-2) = 2(n+1) - 2r'$.
- Case 4: This case is actually impossible if the original polyomino is simply-connected by Theorem 12.

□



(a) Case 1. (b) Case 2.

(c) Case 3. (d) Case 4.

Figure 16: The cases of Theorem 15. The blue cells are from the old polyomino, and the yellow cell is added to form the new polyomino.

The proof is also easily derived from Pick's Theorem (Theorem 14), by noting that for polyominoes, $n = A$ and $b = p$ (Williams and Thompson, 2008). This theorem gives us another proof that the perimeter is even as was proved in Theorem 5. The following is an extension of the theorem above to polyominoes with holes.

Theorem 16. *For a polyomino with H holes, the perimeter is given by:*

$$p = 2n - 2r - 2H + 2.$$

[Not referenced]

Proof. Let P' be the polyomino with all the holes filled.

Let p_0 denote the outer perimeter, and p_i the perimeter around hole i. Let n_0 denote the number of cells in the polyomino plus all the cells in the holes, and n_i the cells in hole i. Let r_0 be the number of interior vertices of P', and r_i the interior vertices of hole i.

The number of vertices on the perimeter of a polyomino is the same as the length of the perimeter of the polyomino. Therefor $r = r_0 - \sum_{i=1}^{H}(r_i + p_i)$. We also have $n = n_0 - \sum_{i=1}^{H} n_i$ and $p = p_0 + \sum_{i=1}^{H} p_i$, and $p_i = 2n_i - 2r_2 + 2$ by Theorem 15.

Let us expand $p_0 - \sum_{i=1}^{H} p_i$:

$$
\begin{aligned}
2n-2r-2H+2 &= 2(n_0-\sum n_i)-2(r_0-\sum(r_i+p_i))-\sum 2+2 && (2.1)\\
&= (2n_0-2r_0+2)+\sum[-2n_i+2r_i-2+2p_i] && (2.2)\\
&= p_0+\sum[-p_i+2p_i] && (2.3)\\
&= p_0+\sum p_i && (2.4)\\
&= p. && (2.5)
\end{aligned}
$$

□

Problem 6.

(1) *Do any of the polyominoes with n cells and minimum perimeter have holes?*

(2) *Can you characterize the minimum perimeter polyominoes?*

See also Problem 10.

Theorem 17 (The perimeter criterion)**.** *Suppose we have a region with mountains of length $e_1, e_2, \ldots$ and a set of tiles with mountains of length $e'_1, e'_2, \ldots$. A necessary condition for the region to be tileable is that each mountain of length e_i, we have $e_i = \sum_j n_j e'_j$ for some $n_i \geq 0$.*

[Referenced on pages 154, 155, 156 and 158]

Example 4. *The 9×9 square with the center removed cannot be tiled by 2×2 squares. The area of the region is 80, and so it satisfies the area criterion. However, we cannot express the edge of length 9 as a multiple of 2, and so it fails the perimeter criterion. In fact, we cannot tile a 9×9 square with a cell removed from anywhere, since we always have at least two borders of odd length. Surprisingly, the minimum number of monominoes necessary to to tile a 9×9 square is actually 17! (See Theorem 173.)*

Theorem 18 (The row-criterion)**.** *Suppose our region rows of length e_i and our tiles have rows of length e'_i. If no rotation is allowed, then a tiling can exist only if $e_i = \sum_j n_j e'_j$ for some $n_i \geq 0$.*

[Referenced on pages 156 and 158]

Example 5. *The tile in Figure 5 can only tile regions with all rows and columns containing an even number of cells. (In fact, if a row or column is not connected, each piece must have an even number of cells.)*

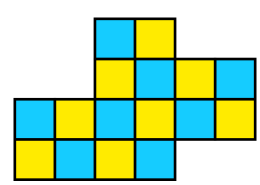


Figure 17: A tile that can only tile regions that have an even number of rows and columns.

The following fact is used to show that if we rotate a polyomino 90°around a point and the rotated copy does not intersect the original, then a copy rotated 180° also does not intersect the original.

(Later, we use this to put constraints on how certain polyominoes can fit into rectangles.)

This fact seems reasonable, but requires some ideas from topology to prove, and therefor I omit the proof.

Theorem 19. *Suppose P and Q are points such that P is Q rotated about $180°$ around O, and there is a continuous path p between P and Q. Suppose we rotate p $90°$ around O to form a new path p'. Then p and p' intersect.*

[Referenced on pages 23 and 24]

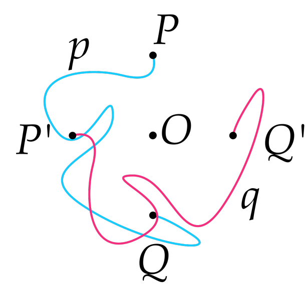


Figure 18: A curve between two points rotated intersects itself under certain conditions.

For a proof, see von Eitzen (2019). The Further Reading section has some references to related ideas.

Theorem 20. *Suppose that P_0 is a polyomino, and we rotate it about any point $90°$, $180°$ and $270°$degrees to get P_1, P_2 and P_3. If $P_0 \cap P_1$ is empty, then $P_i \cap P_j$ is empty for all $i \neq j$.*

[Referenced on page 169]

Proof. If P_0 does not overlap P_1, then by symmetry P_1 does not overlap P_2; P_2 does not overlap P_3; and P_3 does not overlap P_0. We will show P_0 does not overalp P_2; the remaining cases follow by symmetry.

Let's assign coordinates such that the center of rotation is at $(0,0)$, one cell is a unit square, with its edges aligned to the axes.

Note, the center of rotation can fall on either a cell center, or a vertex; it does not affect the proof.

Also note that if the center of rotation falls on a cell, that cell cannot be part of P_0, since otherwise it will also be part of P_1, which contradicts the hypothesis.

Now suppose that P_0 overlaps P_2. This means there is are two cells (x,y) and $(-x,-y)$ that are part of P_0. Since P_0 is connected, there is a path from (x,y) to $(-x,-y)$. By Theorem 19 this path must intersect with a rotated copy, which means P_0 must intersect P_1, which is impossible. Therefore, P_0 cannot overlap P_2. □

2.4 *Further Reading*

Golomb (1996) and Martin (1991) are the classical references on polyominoes; both focus on their recreational aspects and do not require much mathematical background. Toth et al. (2017, Chapter 15) gives a broad overview of polyominoes, and Soifer (2010, Chapter 1) discusses several problem-solving techniques and contains various polyomino tiling exercises[10]. Guttmann (2009) is an advanced book, and deals with the types of problems that have applications in physics.

[10] Chapter 5 is a note that also deals with a polyomino tiling problem.

Winslow (2018) gives a list of interesting open problems concerning polyominoes.

There is a substantial number of web sites dealing with polyominoes. Below are the ones I think are most useful for reference purposes; they all contain lots of data.

- Grekov lists the number of known tilings of rectangles by various polyomino tile sets.
- Reid contains lists of prime rectangles for polyominoes that tile rectangles.
- Dahlke contains some tilings of rectangles by various polyominoes, as well as analysis for polyominoes that don't tile rectangles.
- Myers contains lists of the number of polyominoes that can tile the plane in various ways.
- Oliveira e Silva (2015) contains lists of counts of polyominoes, broken down by the area of their holes and symmetries.

The magnificent Grünbaum and Shephard (1987) covers the general topic of tiling in depth. For a lighter overview, see Ardila and Stanley (2010). The lecture notes Bassino et al. (2015) covers many topics using ideas similar to the ones in these essays. There are more references for general tiling in Section 7.7.

For more on rectilinear polygons, see O'rourke (1987) and Preparata and Shamos (2012). For more on Pick's Theorem, see Varberg (1985) and Grünbaum and Shephard (1993). For a more extensive list of references on Pick's Theorem, see Dubeau and Labbé (2007, Section 1). A very nice application of Pick's Theorem to solve a polyomino compatibility problem is given in Sayranov. (Polyominoes are compatible if they can tile the same region).

I asked for a proof of Theorem 19 on Mathematics StackExchange (see Tulleken (2019)). This question prompted and uncovered several related questions. See YiFan (2019), Gilhooley (2019a), Gilhooley (2019b) and Tremblay (2019).

If you read Latvian, then Cibulis (2001a) and Cibulis (2001b) contains a large number of pentominoes puzzles with solutions and related information.

2.5 *Hexominoes, and Heptominoes*

This section shows the hexominoes and heptominoes, and is included for reference purposes.

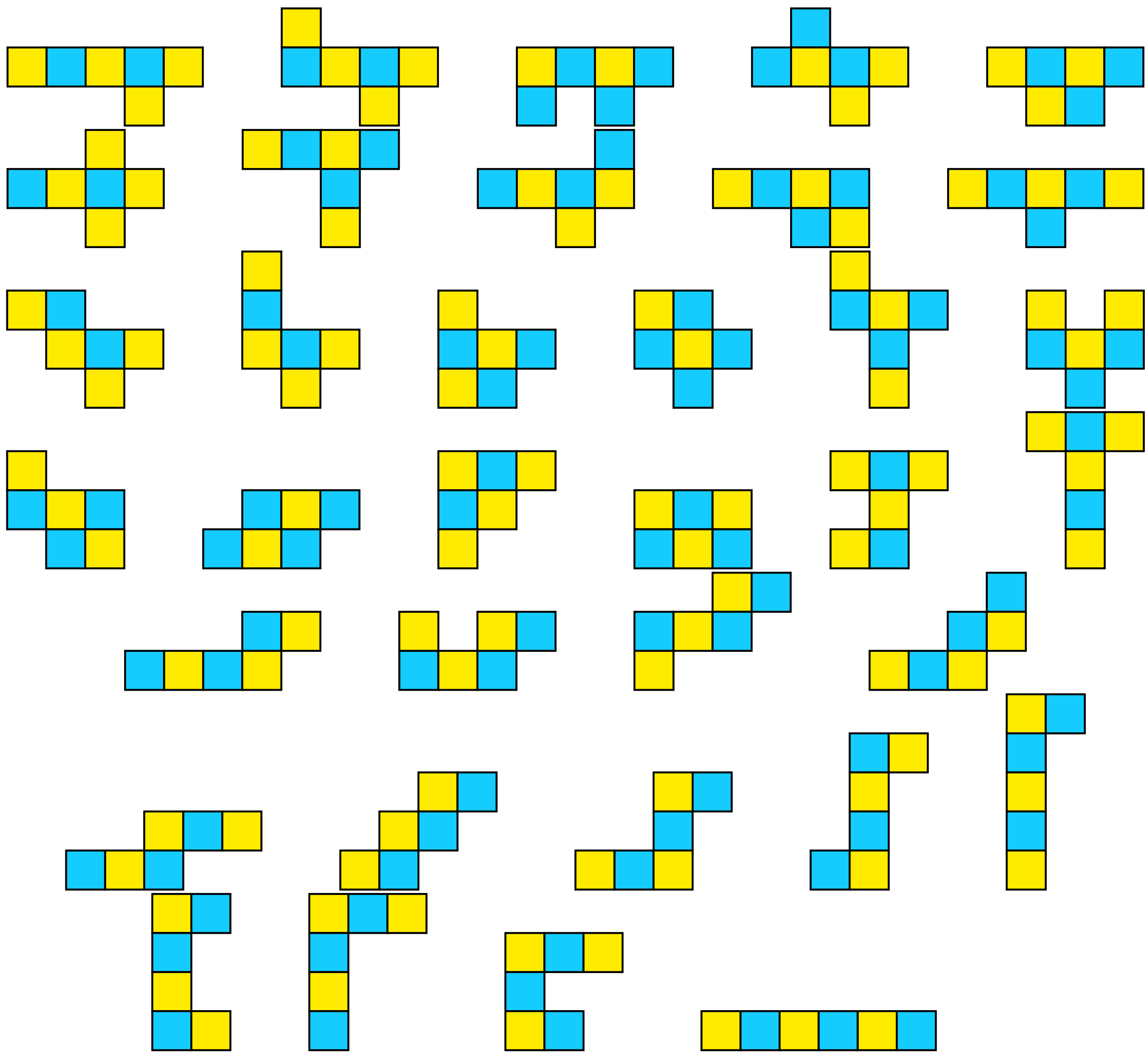


Figure 19: Hexominoes

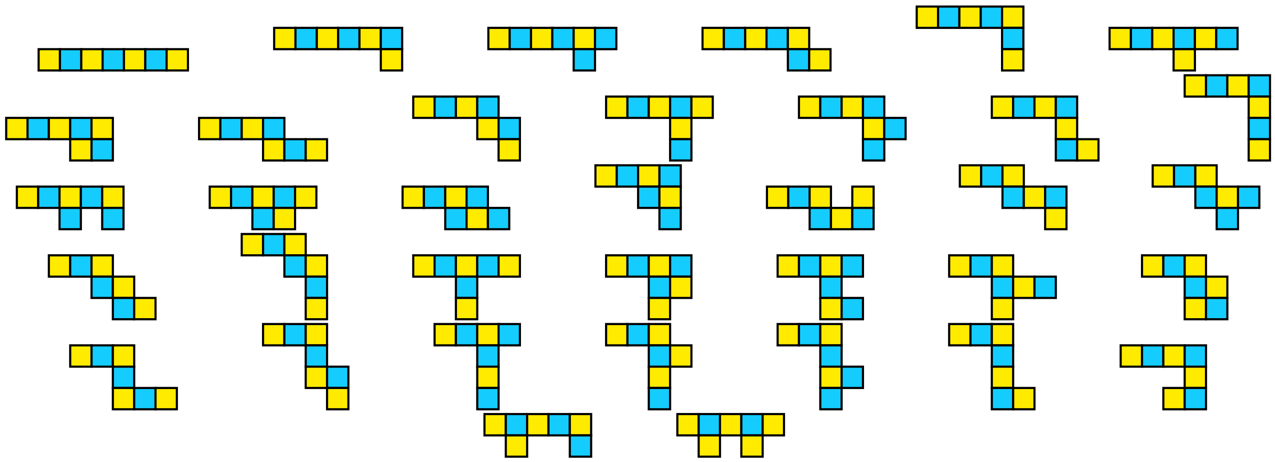


Figure 20: Heptominoes

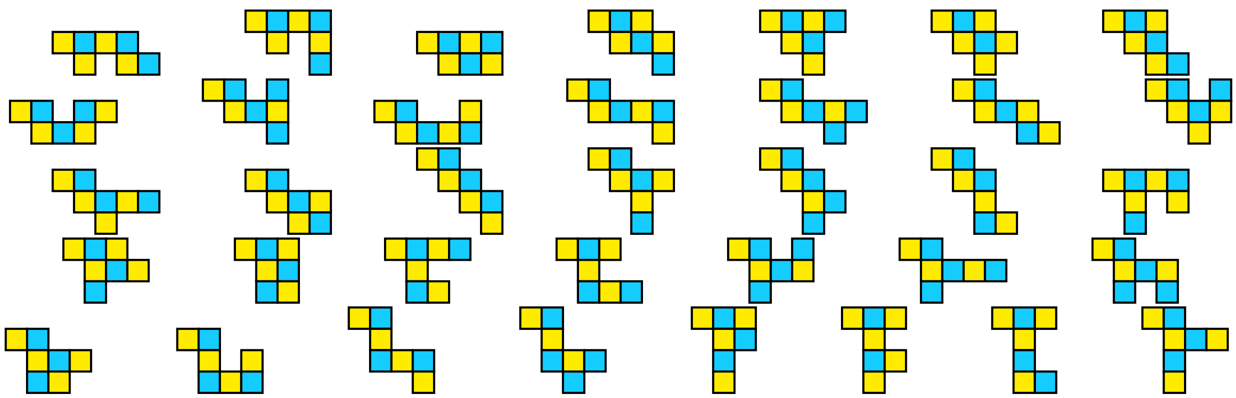


Figure 21: Heptominoes (Continued)

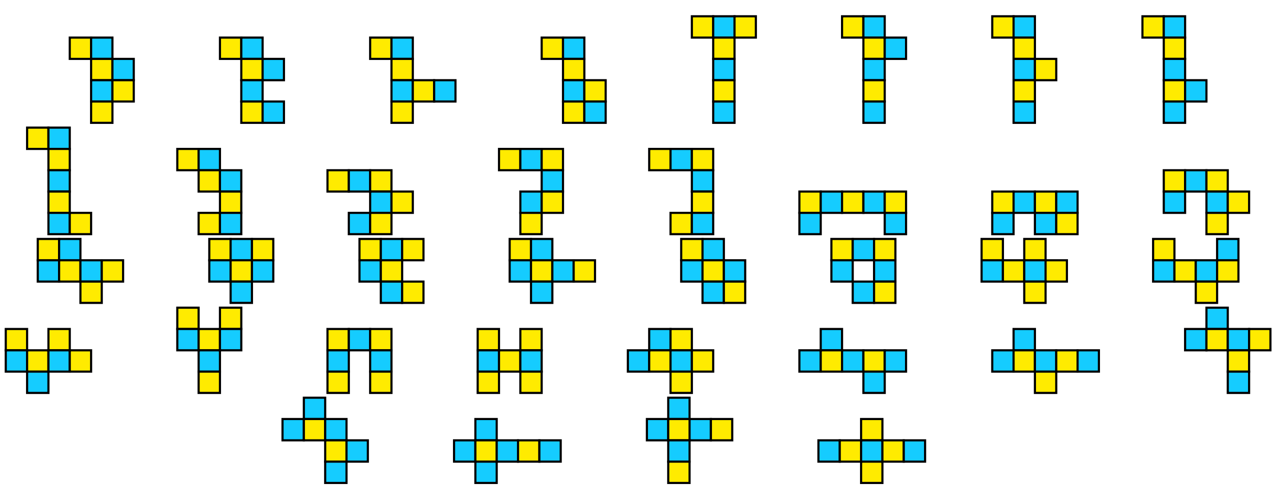


Figure 22: Heptominoes (Continued)

3
Dominoes I

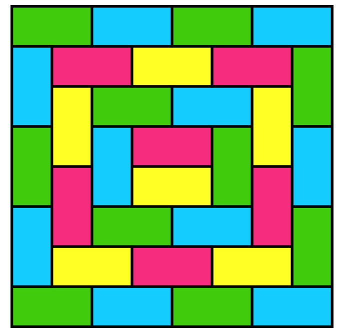


Figure 23: A tiling of a 8×8 square by dominoes.

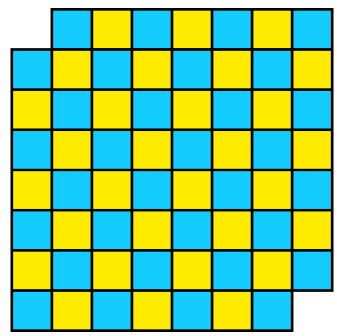


Figure 24: The mutilated chessboard.

FIGURE 27 SHOWS some examples of regions that cannot be tiled by dominoes. That some regions cannot be tiled by dominoes is intriguing. How can shapes as small as dominoes not be arranged to fit into a region as big as in Figure 27(e), when there are so many options?

This is the topic of the first section: understanding why some regions have a tiling by dominoes and others don't.

Figure 68 provides us with another mystery. In each region, there are dominoes that fit into the tiling in just one way (they are marked in yellow). In the first region it is easy to see why this must be so, but what is happening in the last region? How is it not possible to find a tiling so that at least some of those dominoes lie in a different position?

This is the topic of the second section: understanding how the same region can be tiled in different ways and how the region can force dominoes into certain positions.

A FAMOUS PROBLEM, the *mutilated chessboard problem*[1], illustrates some key ideas from each section.

Consider an 8×8 chessboard, with two opposite corners removed as in Figure 24. Can the board be tiled? If you know this problem, you know that it cannot. Because, every domino must cover exactly one black and one white square, and with opposite corners removed, there are more cells of one color than the other, and so a tiling is impossible. Of course, this principle can be applied to any region, and gives us a valuable tiling criterion. We will use this as the basis for constructing tiling criteria more powerful than the area criterion (Theorem 1) that we gave in the introduction.

If we remove two squares of opposite colors instead, is a tiling always possible? The answer is, *yes*, a tiling is always possible. Here they key is to notice that we can divide the board into two strips, as

[1] The mutilated chessboard problem, was first proposed by Max Black in Black (1947), and has been discussed in various places, including (Golomb, 1996, p. 4), (Martin, 1991, p. 1-4, 7-9), (Mendelsohn, 2004) and Engel (1998).

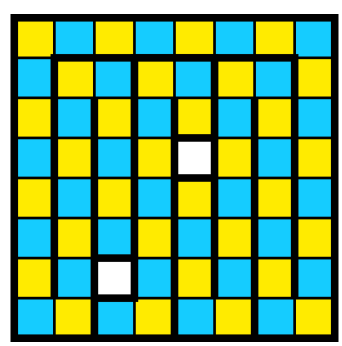


Figure 25: Dividing the chessboard into two strips

shown in Figure 25. And because the cells have different colors, no matter where we remove the two cells from, the end-points of each strip must have different colors, and so it has an even number of cells. Each of these strips has an even number of cells, and is tileable in the obvious way. (We will go over this logic in more detail once we made some proper definitions.)

One way to show a region is tileable, is to partition it into strips with an even number of cells. If the end point of a strip of more than two cells is next to its starting point, we say the strip is closed. Closed strips have at least two tilings, and this is the basis of the second section. We will see that cells that can only be tiled one way can never be part of a closed strip, and that all tilings of a region can be obtained from the two tilings of each closed strip in it.

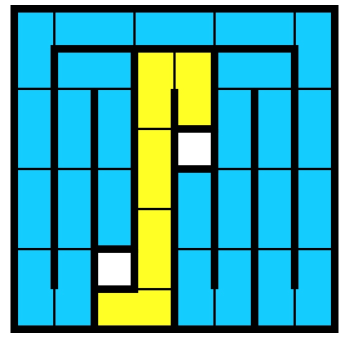


Figure 26: A tiling along the strips.

3.1 *Tiling Criteria*

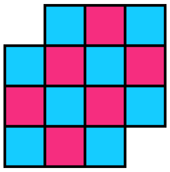
IN THIS SECTION we develop some tools with which we can tell whether a region is tilable by dominoes or not.

We could, of course, use brute force to check all the regions in Figure 27 (or get a computer to do it). There are two reasons to look for something better:

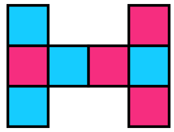
- So that we can write faster programs. While a naive program might deal with all our examples in seconds, scientists using tilings to understand how molecules fit in solids need something better. Their "tiling problems" may be regions with millions of cells!
- So that we can understand how these tilings work, and understand other related phenomena. Section 3.2 gives a taste of this.

Our goal is to develop four techniques:

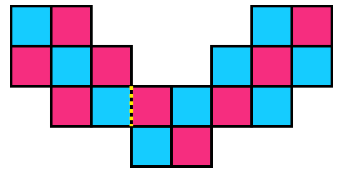
- A flow criterion: a technique of coloring cells in a region and analyzing the counts of dominoes that must cross the borders of subregions.
- Cylinder reduction: a geometric transformation that preserves the tileability of a region and can be used to simplify regions.
- The marriage theorem: another way to determine if a tiling exists.
- A generalized way of coloring regions to exhibit their untiliablity.



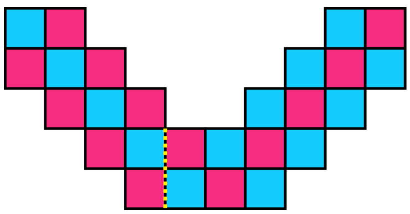
(a) An example of an unbalanced polyomino.



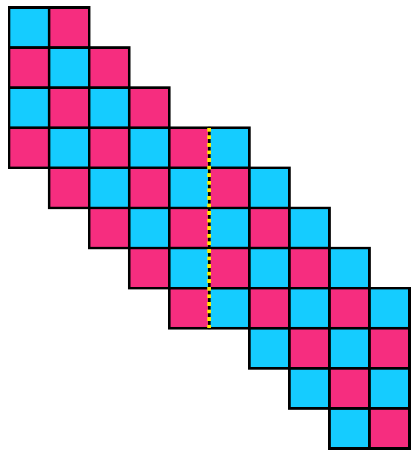
(b) An example of a balanced non-compact polyomino that cannot be tiled.



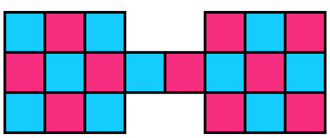
(c) An example of a balanced compact shape that cannot be tiled.



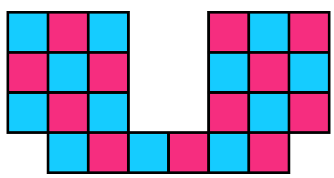
(d)



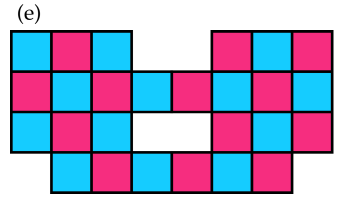
(e)



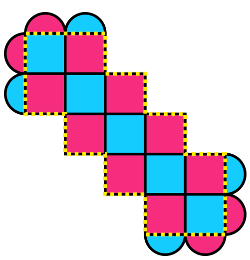
(f)



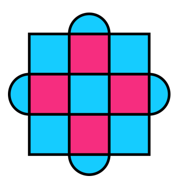
(g)



(h)



(i) No matter how this region and the next one are extended at the blobs, the resulting regions will not be tileable



(j)

Figure 27: Regions that cannot be tiled with dominoes.

We do not arrive at the state-of-the-art in solving domino tiling problems (just yet), however, the tools will help us deal with all the regions in our examples in Figure 27, and give us an intuitive understanding of how to tackle tiling problems.

A key point to understand from this section is: the border of a region is important, and tell us a lot about how a region can be tiled.

3.1.1 Flow

SINCE ALL TILINGS MUST SATISFY the area criterion (Theorem 1), we know a tiling by dominoes can exist only if the area of the region we wish to tile is even. Not all regions with even area are tileable (all the regions in Figure 27 have even area).

Now suppose we divide a region into two parts, each with an odd area. If the original region is tileable, then we know that there must be at least one domino that lies in both of the two parts. In fact, we know the number of dominoes that crosses the border between the two parts must be odd, otherwise the remaining regions will have odd area and not be tileable. Figure 28 illustrates this principle.

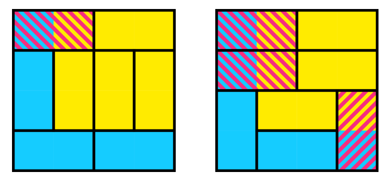


Figure 28: Two examples of the tiling of a square. The square is divided into two partitions (yellow and blue). Since both partitions have odd area, any tiling such as the two shown above must have an odd number of dominoes that cross the border.

Theorem 21 (Border Crossings Theorem)**.** *In a subregion S of a region with a tiling, the number of dominoes that cross the border of the subregion must have the same parity*[2] *as the area of the subregion.*

[Referenced on pages 31, 34, 53, 91, 112 and 119]

[2] Even or odd.

Proof. Let k be the number of dominoes that cross the border of S. Each of these dominoes has only one cell in S. If we remove these cells to form a new region S', we have a region that is tileable, with area $|S'| = |S| - k$. Since S' is tileable, $|S'|$ must be even (by Theorem 1), and so $|S|$ and k must either both be odd, or both be even. $\square$

Below we give two applications of this theorem; as a theorem and an example.

A *bridge* is a sequence of cells in a region such that each has exactly two neighbors, and such that removing any cell in the bridge will leave the region disconnected. For example, Figure 27(f) and (g) each has a bridge, Figure 27(h) does not (since no cell splits the region when removed). Figure 27(c) also does not have a bridge; although there are single cells that will split the region if removed, they have more than two neighbors.

Theorem 22. *Each cell in a bridge can be tiled in only one way.* [3]

[3] Such cells are called *frozen*. We will define this concept in the next section.

[Not referenced]

Proof. Let R be a region with a bridge, and let v be a cell in that bridge. Because v is a cell in a bridge, it has exactly two neighbors– let's call them u and w. Further, $R - \{v\}$ is disconnected. Let S_u and S_w be the two disconnected subregions that contains u and w respectively. The border of each of these share exactly one edge with the cell v, in particular, S_u shares a border with v at the edge between u and v.

Now consider S_u. If $|S_u|$ is odd, then the number of dominoes that crosses the border of S_u is odd. The only place where a domino can cross is for a domino to cover both u and v, and therefor any tiling of R must have a domino in this position.

If on the other hand $S|S_u|$ is even, then the number of dominoes that crosses the border of S_u is even. Since there is only one place where a domino can cross, it means the number of dominoes that cross must be 0. Therefor, a single domino cannot cover both u and v, and so, a single domino must cover v and w. Therefor, every tiling of R must have a domino in this position.

Taken these together, v can only be tiled one way. The same argument applies to all oither cells in the bridge, and therefor the bridge has a unique tiling in R. □

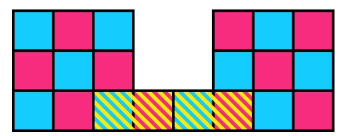


Figure 29: The cells marked yellow can be tiled in only one way.

The next example uses Theorem 21 in a more sophisticated way to prove that a tiling does not exist. [4]

[4] The author mentions this idea has been known before.

Example 6 ((Mendelsohn, 2004)). *Let's look at the mutilated chessboard. The top row has an odd area. Therefore, an odd number of dominoes must cross its border (Theorem 21), and only vertical dominoes that also lie in the second row can do that (otherwise, parts of dominoes will fall outside the mutilated chessboard).*

The second row has even area, and so an even number of dominoes must cross its border. We already have an odd number of dominoes that cross the border from the top, therefore we must have an odd number of vertical dominoes that cross the border to the bottom.

Following the same argument, we get that there must lie an odd number of vertical dominoes between each pair of adjacent rows. There are 7 such pairs, so we can conclude the total number of vertical dominoes must be odd.

Applying this idea to the columns, we find there must be an odd number of horizontal dominoes too. So the total number of dominoes must be even. But to cover the 62 squares we need 31 dominoes, which is odd. And therefore, a tiling is impossible.

This example shows that we can always determine the parity of the number of vertical and horizontal dominoes in a tiling. For a tiling to exist, this must be consistent with the parity of the total number of dominoes (which we can deduce from the area).

SATISFYING THE AREA CRITERION is not enough (all the regions in Figure 27(a)-(h) are untileable and have even area), and checking the parity as in the previous example is also not enough (for example, Figure 27(f)).

We will now look at the color argument we discussed at the beginning of the chapter, and see how far it gets us. The **checkerboard coloring** plays an important role in our discussions going forward, and to make it easier to talk about and prove details we introduce some additional notation and terminology.

In a region R with checkerboard coloring applied, let $\mathcal{W}(R)$ denote the white cells in R, and let $\mathcal{B}(R)$ denote the black cells in R. The **deficiency**[5] of R is defined as

$$\Delta(R) = |\mathcal{B}(R)| - |\mathcal{W}(R)|. \tag{3.1}$$

If the deficiency of a region is 0, the region is **balanced**.[6]

The functions $\mathcal{W}$, $\mathcal{B}$ and Δ depend on which of two ways the image has been colored. However, the absolute value of the deficiency and being balanced are inherent features of a region, and are independent of the coloring used.

[5] Also called *bias*.

[6] The term *balanced* is used informally in (Golomb, 1966, p. 17). The term , as well as the white and black functions, are defined explicitly in for example Thiant (2003). The term is also used in Herzog et al. (2015) for a completely different concept. The words *biased* and *unbiased* is sometimes used instead of unbalanced and balanced, for example Mason (2014).

Theorem 23 (Checkerboard Criterion)**.** *For a region to be tileable by dominoes, it must be balanced (Golomb, 1996, p. 4).*

[Referenced on pages 41, 45, 46, 71, 106, 107 and 119]

Problem 7.

(1) *Prove that* $\Delta(R) \equiv |R| \pmod 2$. *Note that this imploes that the area criterion is redundant.*

(2) *Show that* $\Delta(R(m,n))$ *equals* 0 *or* 1 *depending on whether the area of the rectangle is odd or even.*

(3) *Prove* $|\Delta(R)| \leq |R|$.

(4) *Prove that if R is partitioned into two subregions S_1 and S_2, then* $|\Delta(R)| \leq |\Delta(S_1)| + |\Delta(S_2)|$.

Using this criterion, we can prove Figure 27(a) is not tileable. However, it *still* is not enough, since all the regions in Figure 27(b)-(h) are balanced and untileable.

Problem 8. *Find examples of non-tileable balanced regions.*

What is the largest the deficiency can be? Figure 30 shows we can make the deficiency as large as we want by extending it as shown. However, it is bounded by the number of cells, as stated in the following theorem.

Theorem 24. *The deficiency of a connected region R is bounded by the number of cells as follows:*

$$|\Delta(R)| \leq \frac{|R|+1}{2}.$$

[Not referenced]

Proof. Let $B = \mathcal{B}(R)$ and $W = \mathcal{W}(R)$. WLG, assume that $B > W$, so $\Delta(R) > 0$. We will shows that $B \leq 3W+1$, from which the result follows. Because if $B \leq 3W+1$, we have $2B - 2W \leq W + B + 1$, that is, $2\Delta(R) \leq |R| + 1$, or $\Delta(R) \leq \frac{|R|+1}{2}$.

We now prove $B \leq 3W+1$. Suppose not; that is, suppose $B > 3W+1$. Then there must be at least one white cell with 4 black neighbors, with none of these 4 black cells having any other white neighbors. If there are no other cells than these 5 cells, we have $4 = B = 3W+1$, so there must be more. But then these 5 cells cannot be connected to any other cells, and we have a disconnected region which contradicts our hypothesis that the region is connected. □

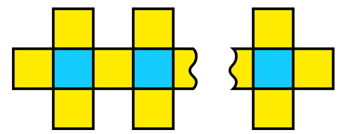


Figure 30: A family of regions that obtain maximum deficiency.

This upper bound is achievable by regions like the one shown in Figure 30. A polyomino that achieves the maximum amount of bias possible for its area is called **maximally biased** (Mason, 2014). The number of maximally biased polyominoes is shown in Table 4.

n	A234013
1	1
2	1
3	2
4	1
5	1
6	11
7	8
8	3
9	1

Table 4: The number of maximally biased polyominoes.

$\vert R\vert$	$\vert\Delta(R)\vert$
$4k$	$2k$
$4k+1$	$2k+1$
$4k+2$	$2k$
$4k+3$	$2k+1$

Table 5: Maximum deficiency.

Problem 9. *(Mason, 2014) Show the following facts about $4n+1$-sized biased polyominoes, with $\mathcal{B}(R) > \mathcal{W}(R)$:*

- *They are all of made from a monomino and T-tetrominoes like the one shown in Figure 30.*
- *The removal of any white square will result in an illegal, disconnected polyomino.*
- *The same for any twice-connected black square.*
- *The removal of any once-connected black square will result in a legal, maximally biased polyomino of size $4n$.*
- *The addition of a white square will result in a maximally biased $(4n+2)-$polyomino;*

Not all maximally biased $(4n+2)$-polyominoes can be generated from $(4n+1)$-maximally biased polyominoes, as shown in Figure 31 (Mason, 2014).

Problem 10.

(1) *Show that the maximum perimeter for a polyomino with area n is $2n+2$. (See also Theorem 15).*

(2) *Show that the number of polyominoes with area n that have a perimeter of $2n+2$ (**A131482**) is the number of maximally biased polyominoes for $n = 4k+1$.*

See also Problem 6.

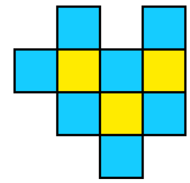


Figure 31: A maximally biased polyomino that cannot be formed by extending a maximally biased polyomino with one less cell.

In the same way that we turned the area criterion into something more useful by looking at what must happen at the border of subregions, we now turn the color criterion into something more powerful by looking at the border of subregions.

Let R be a region with the checkerboard coloring applied, and let S be a subregion of R. If w is the number of dominoes covering the border of S with a white cell inside S, and b is the number of dominoes covering the border with their black cells inside R, then we define the **flow**[7] of S as

$$\phi(S) = b - w. \tag{3.2}$$

[7] This definition is essentially given in Saldanha et al. (1995).

See Figure 32.

When $S = R$, the flow is 0, since there are no dominoes that cross the border, and so $w = b = 0$.

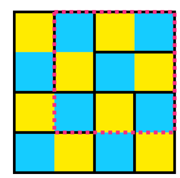


Figure 32: Let S be the region inside the red border. Then $b = 2$, $w = 1$, and so $\phi(S) = 2 - 1 = 1$. Also, $\mathcal{B}(S) = 5$, $\mathcal{W}(S) = 4$, and so $\Delta(S) = 5 - 4 = 1$.

Theorem 25 (The Flow Theorem)**.** *Let S be a subregion of R, and suppose R has a tiling by dominoes. Then the flow of S equals the deficiency of S, that is,*

$$\phi(S) = \Delta(S). \tag{3.3}$$

[Referenced on pages 35, 53, 54 and 119]

Proof. Let W (B) be the number of white (black) cells inside S, let w (b) be the number of dominoes that cross the border with their white (black) cells inside S, and let k be the number of dominoes completely in S. Then $B - W = (b+k) - (w+k)$, and so $B - W = b - w$, and thus $\Delta(S) = B - W = b - w = \phi(S)$.

□

Note the similarities between border crossings theorem (Theorem 21) and the flow theorem above. Both give us information about what happens at the border based on what is inside the region. But the flow theorem gives us much more information; Theorem 21 merely tells us the parity of the number of dominoes that cross the border. The flow theorem gives us the difference of the two different types of dominoes.

Also note that the area must have the same parity as the flow: if the area is even, then W and B are both odd or both even. In either case, their difference is even. This means with the flow theorem in place, the border crossing theorem is now redundant. [8]

[8] One reason to differentiate the two theorems is that the border crossings theorem is easier to generalize to other tile sets.

The following examples show how to use the flow theorem to prove a region is not tileable.

Example 7. *Consider Figure 27(c), and let's choose the subregion as the shape left of the dotted line. Now since $W = 3$ and $B = 4$, we have $|W - B| = 1$, so we know a domino must cross the dotted line. Removing this domino partitions the shape into two shapes, each of which is untileable because they don't satisfy the area criterion.*

Example 8. *Consider Figure 27(d), and choose the subregion as the region the right of the dotted line. We have $|B - W| = 2$, which implies dominoes must overlap the dotted line in two places. But this means $b = w = 1$, and so $|b - w| = 0$, which is impossible if the region is tileable. Therefore, it is not tileable.*

Example 9. *Consider Figure 27(e). Partition it in halves by a vertical cut through the middle. Then $|B - W| = 4$, but the maximum value that $|b - w|$ can have is 3. Therefore, the region is not tileable.*

Example 10. *Consider Figure 27(i), and consider the colored subregion. The deficiency $|B - W| = |8 - 5| = 3$. This means, at least three dominoes must overlap the border. However this is done, we are always left with a region with 4 black and 3 white squares, which is untileable. Therefore, the entire region is untileable.*

Problem 11. *If we have a domino tiling iof a region, show that we can determine the parity of vertical dominoes with their top squares black. (Follow the type of reasoning used in Example 6.)*

If you tried your hand at Problem 8 and followed the examples above, you may have noticed that you can create an untileable region by having an abundance of black on the one side of the region, and a abundance of white on the other side, with a choke point between the two parts. The following theorem gives a formulation of this idea.

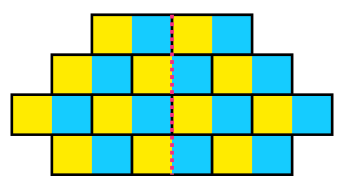


Figure 33: The deficiency of the left partition is -2, so the length of a cut through the center must be at least 3.

Theorem 26. [9] *Suppose we apply the checkerboard coloring to a tileable region, and partition it into two subregions with a straight cut. If one subregion has W white cells and B black cells, then the cut must have length at least $2|B - W| - 1$.*

[9] In Kenyon (2000c) the author mentions that a very similar theorem has been proven in Fournier (1996) (in French).

[Referenced on page 70]

Proof. Assume $B \geq W$. From the flow theorem (Theorem 25), we know that the number of tiles that cross the cut must be at least $B -$

$W = b - w$. From this we get $b = B - W + w$, so $b \geq B - W$ (since w is non-negative). So the number of dominoes that cross the cut with a white cell inside the subregion must be at least $B - W$. Along a straight cut, there are at most $\left\lceil \frac{L}{2} \right\rceil$ places where this can happen, so it needs to have length at least $2(B - W) - 1$.

If we assume $W \geq B$, we can show the length of the cut must be at least $2(W - B) - 1$ following the same argument as above, reversing the roles of black and white.

Putting these together, we arrive at the result: the length of the cut is at least $2|W - B| - 1$. □

See Figure 33 for an example.

Problem 12. *What if the cut is not straight?*

You may also have noticed that to create unbalanced regions or parts of regions, you have to manipulate the border of the region so you have a lot of corners of the same color. Also, we have already seen some theorems that relate the border of a region to what is going on inside. The following theorem shows we can determine the deficiency from what is going on at the border alone.

Theorem 27. *Let b be the number of black edges on the border, and w be the number of white edges on the border. Let B be the number of black squares and W be the number of white squares. Then $b - w = 4(B - W) = 4\Delta(R)$.*

[Referenced on pages 37, 70, 71 and 115]

Proof. Consider building a region cell by cell. At each stage, we can either add a white or a black square. If we add a white square, it is a neighbor of 0, 1, 2, 3, or 4 other cells, all of which must be black. Each exposed edge must be white, and each unexposed edge must reduce the total number of black edges. So the total amount that $b - w$ decreases is 4. A similar argument shows that if we add a black square, $b - w$ is increased by four. In other words: $b - w = 4(B - W)$. □

It follows that $B = W$ if and only if $b = w$.

Problem 13. *A corner cell is a cell with two adjacent edges that are not shared with other cells. Let R be a region such that each of its corner cells have no neighbors that are on the border (that is, all neighbors of each corner cell are interior cells). Prove R is untilable.*

Theorem 28 (Csizmadia et al. (1999), Theorem 2.1, Theorem 3.1). *If all n edges of a simply-connected region R have odd length,*

(1) *the deficiency of R is given by*

$$\Delta(R) = \frac{n}{4},$$

(2) *the region is unbalanced, and*

(3) *the region is untilable by dominoes.*

[Referenced on page 37]

Proof. [10] Suppose one corner edge is black. Then all corner edges are black. This means the ends of sides are all black, and so for each side i, we have:

$$B_i - W_i = 1,$$

and summing over all sides:

$$\sum_{i=1}^{n}(B_i - W_i) = n.$$

But $\Delta(R) = \frac{\sum_{i=1}^{n}(B_i - W_i)}{4}$ (by Theorem 27), and thus $\Delta(R) = \frac{n}{4} > 0$, and so R is unbalanced, and hence untilable by dominoes. Note we already proved that n is divisible by 4 in Theorem 6. □

[10] The proof here is new. The proofs in Csizmadia et al. (1999) is quite complicated.

The theorem does not hold for regions with holes. For example, the 3×3 square with its center removed has all its sides odd, but it is balanced and even tileable (Figure 34).

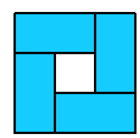


Figure 34: A region with a hole that has all its edges with odd length that is balanced and tileable by dominoes.

Theorem 29. *A balanced polyomino must have at least two sides of even length.*

[Referenced on page 119]

Proof. For the polyomino to be balanced, it must at least have one even side (Theorem 28). We also know the perimeter must be even (Theorem 5). Therefor, the number of odd sides must be even. But the total number of sides must be even (Theorem 7), and so the number of even sides must *also* be even. Thus, there must be two or more even sides. □

Theorem 30. *A polyomino with all sides even is tileable. (Mentioned in Kenyon, 2000b, Section 8).*

[Referenced on page 119]

Proof.

(1) For regions with no holes:

(a) If the figure has exactly four knobs, it must be a rectangle. In this case we can tile the region with horizontal dominoes only.

(b) If the figure has more than four knobs, we can remove one. Suppose a knob B lies between sides A and C, with $A \leq C$. We can remove a rectangle S_1 with sides of length B and A, and since both are even we can tile S_1 with horizontal dominoes. We can repeated this process until a single rectangle is left, which we can tile as we did above.

Note that the long edge of a domino is shared with at most one domino.

(2) For regions with holes:

(a) First tile the filled polyomino (i.e. the polyomino with all holes filled) with horizontal dominoes as in the procedure above.

(b) Remove all dominoes that lie completely inside a hole.

(c) The ones that lie partially in a hole come in pairs, since all sides are even. Each pair can be replaced with a single domino that covers only the cells that do not lie in the hole.

This completes the tiling.

□

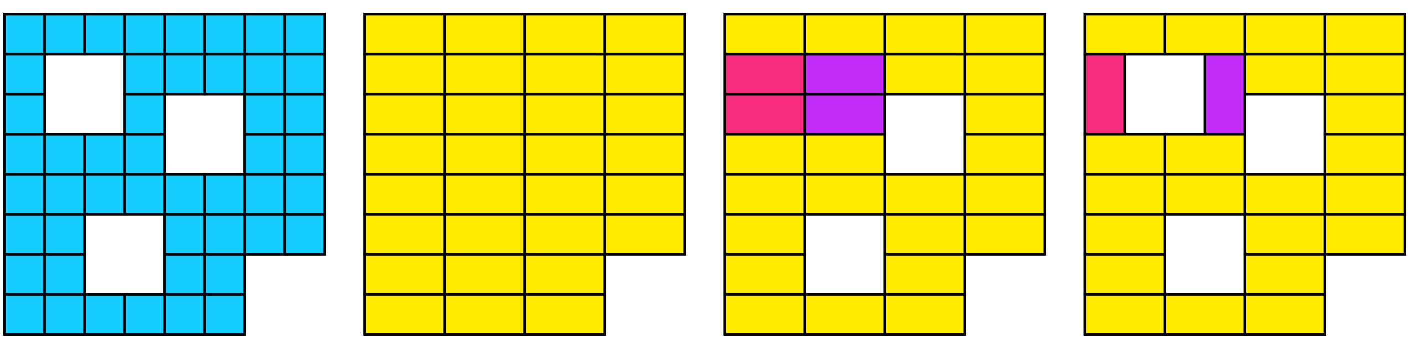


Figure 35: An example showing the procedure in the proof.

Problem 14.

(1) *Can you modify the theorem's conditions to be weaker?*

(2) *Prove that if the sides of a region is divisible by n, it can be tiled by bars of length n if holes are at least $n-1$ units apart.*

(3) *If we use all n-ominoes for the tileset, can holes be anywhere?*

3.1.2 *The Marriage Theorem*

INFORMALLY, THE *marriage theorem* states the following[11]: Suppose we have a group of k white cells from a region R, and they have n black neighbors in R. If $n < k$, then no tiling exists. Moreover, if every group of white cells in R has at least as many neighbors as white cells in the group, a tiling exists.

[11] This is a specific version of a much more general theorem given in Hall (1935) that is now called the *marriage theorem*, with applications that go beyond tiling. Covering the more general results fall outside the scope of this essay, but we give some references in the *Further Reading* section. This version is given in Ardila and Stanley (2010).

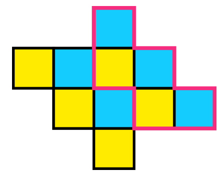


Figure 36: In this region, the cells surrounded by a pink line is a black bad patch. The rest of the region is a white bad patch.

Before we prove it, we need some terminology to make it easier to make our statements.

A subregion S of R is called a **white patch** of R if all the neighbors of its white cells are also in S, and there are no other black cells in S. In other words, all black cells in S have at least one neighbor also in S. A similar definition can be made for **black patch**.

A region itself is a patch. If a patch is not equal to the region we call it a **proper patch**. A white patch can be, but is not necessarily, a black patch. In fact, if R is connected, a white patch is only also a black patch if it is the entire region.

Theorem 31. *Suppose R is connected, and S is both a white patch and a black patch. Then $S = R$.*

[Not referenced]

Proof. Suppose R has some cells that are not in S. Since R is connected, we must have at least one of these cells be a neighbor to a cell in S. Let's call this cell u, and its neighbor in S, v. Suppose v is black, then because S is a black patch, all its neighbors must lie in S, and this contradicts that u lies outside S. And if v was white, because it is a white patch, all its neighbors must lie in S. Therefore, there can be no cells in R that are not also in S, and therefore $S = R$. □

Theorem 32. *Let S be a white patch of a region R. Then $R - S$ is a black patch of R. Similarly, if S is a black patch of R, then $R - S$ is a white patch.*

[Referenced on page 40]

Proof. We only prove the first part; the second part can be proven with the exact same argument with the roles of white and black reversed.

Suppose S is a white patch, but $R - S$ is not a black patch. Then there is a black cell u in $R - S$ with a neighbor v *not* in $R - S$. Then v must lie in S, and it is white. Since S is a white patch, all the neighbors of v, including u must lie in S. We arrive at a contradiction, and so $R - S$ must be a black patch. □

We call a white patch **bad** if it has fewer black than white neighbors, and **good** otherwise. Similarly for black patches.

Theorem 33. *A balanced region has a bad black patch if and only if it has a white bad patch.*

[Referenced on pages 40, 41, 52 and 87]

We only prove the first part, the second part follows from the same argument reversing the roles of white and black.

Proof. *If.* Suppose that S is a bad white patch. Then $R - S$ is a black patch (Theorem 32). Since S has fewer black than white cells and R is balanced, $R - S$ must have fewer white cells than black cells, and so it is bad. □

It follows directly from this that if all white patches in a region are good, then so are all black patches.

Theorem 34 (The Marriage Theorem)**.** *A region is tileable if and only if it has no bad patches.*

[Referenced on pages 42, 43, 52 and 119]

Proof. *If.*[12] Suppose all the white patches of a region are good. It follows that all the black patches of the region must be good too (and vice versa) by Theorem 33. We can now partition the region into two subregions S and $R - S$ in one of two ways:

(1) If all the proper white patches of R are unbalanced, then each patch of R must have strictly more black cells than white cells. Let S be a white cell and its neighbor. Then S is tileable (with a single domino), and $R - S$ is a region with all its white patches good. This follows from the fact that we removed only one black cell, so the number of black cells in each patch can can drop by at most 1, and since these are strictly bigger than the number of white cells, after the drop there must be at least as many black cells as white cells in the patch.

(2) If at least one of the patches in R is balanced, we let S be such a patch. Then S must have all its white patches good, and $R - S$ must have all its black patches good (and so, it must also have all its white patches good). Both partitions must also be balanced.

Note that a patch in S or $R - S$ need not be a patch in R.

[12] The logic of this part of the proof follows the proof in (Kung et al., 2009, p. 56), due to Easterfield (Easterfield, 1946) and Halmos and Vaughan (Halmos and Vaughan, 2009).

We can continue this process, and it must eventually end, since the number of cells in R is finite. And so, we eventually arrive at a bunch of subsets of two cells each, all tileable by a single domino, and so the entire region must be tileable by dominoes.

Only if. If the region is not balanced, we know it is not tileable by Theorem 23.

Suppose then it is balanced, and suppose it has a bad white patch. (If it had a bad black patch, it must also have a bad white patch by Theorem 33, so there is no loss in generality.)

If there is a tiling, each white cell in the white patch has an associated black neighbor that lies in the same domino, and there must be at least as many of these black cells as white cells. However, the patch is bad so this is not the case, a contradiction. Therefore no tiling exist. □

We finally have a criterion that can work for all tilings. However, the problem is that it can be difficult to find a bad patch, or to show there aren't any. For example, in a region R, the flow theorem is easier to apply (although, of course, you *could* find a bad patch.)

That does not mean it is not useful: We will use this theorem a few times to prove some other things about tilings, and it also gives us a way the come up with untileable regions easily as the next theorems show.

An **extension** of a region R is a region P such that R is a subregion of P. Informally, an extension of R is some region formed by adding cells to R.

Theorem 35. *Suppose R is a region with an odd number of cells. If we apply the checkerboard coloring such that $\mathcal{W}(R) < \mathcal{B}(R)$, then any extension of R that adds no new neighbors to black cells in the polyomino is untileable.*

[Not referenced]

Proof. Let the original region be R, and its extension P. Since $W < B$, R is a bad patch with respect to itself. But since we do not add any neighbors to form P, R is also a bad patch with respect to P, and therefor untileable. □

Theorem 36. *If a region has a bad patch, it has a connected bad patch.*

[Not referenced]

Proof. Let S be the bad patch. WLG assume that $\mathcal{B}(S) < \mathcal{W}(S)$. Now partition S into connected disjoint sets S_i such that the (black) neighbors of all the white cells in S_i is also in S_i. We want to show one of these sets is a bad patch. If each S_i is a good patch, then $\mathcal{B}(S_i) \geq$

$\mathcal{W}(S_i)$ for all i. But then $B(s) = \sum_i \mathcal{B}(S_i) \geq \sum_i \mathcal{W}(S_i) = \mathcal{W}(S)$, a contradiction. □

This theorem shows the process of generating untileable regions described above can generate every untileable region.

3.1.3 *Cylinder Deletion*

We can also use the Marriage Theorem (Theorem 34) to prove certain reductions do not affect the tileability of a region.

A **vertical n-cylinder** is a simply-connected region where each row has n cells (see Figure 37). A **horizontal n-cylinder** has n cells in each column (Hochberg, 2015).[13]

[13] Cylinders also play an important role in tiling extensions (see for example Theorem 123) and tilings of the infinite strip.

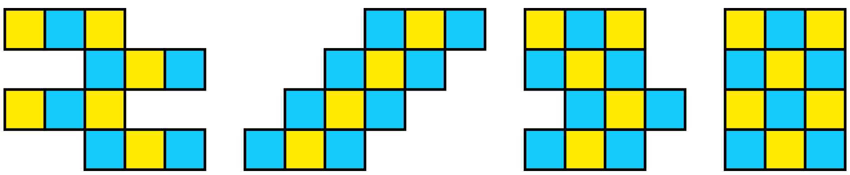
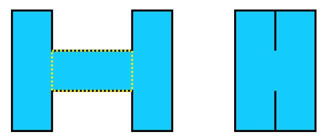


Figure 37: Examples of vertical 3-cylinders.

An n-cylinder is tileable by dominoes if n is even, since each row has an obvious tiling by horizontally-placed dominoes.



(a) (b)

Figure 38: Cylinder deletion. (a) The cylinder S is the shape contained in the yellow dotted line, and R is the entire region. (b) $R \ominus S$. Note the barriers that cannot be crossed by dominoes.

Suppose a vertical cylinder S is a subregion of R such that it shares is top and bottom borders with the border of R. If we remove S from R, and move the two pieces together, we get the new region $R \ominus S$. We call this operation a **deletion**. An analogous definition can be made for a horizontal cylinder S. After a deletion, we my be left with a region that has a barrier. For example, in Figure 38 we delete a cylinder from the double-T region. This yields a new shape with two internal barriers that cannot be crossed by dominoes. In particular, it means the two top cells are not neighbors of each other. Note that we cannot delete a horizontal cylinder from the reduced region because of the barriers.[14]

[14] We will later see how this geometrical operation is equivalent to simplifying the border word algebraically.

The operation is also very similar to deletions on rhombus tilings, also called *contractions*. In this context, the deleted shape is called a *de Bruijn section*. See for example Chavanon and Remila (2006) and Chavanon et al. (2003).

Theorem 37. *Let S be an n-cylinder with n even. Then R is tileable, if $R \ominus S$ is tileable.*

[Referenced on pages 49, 86, 108 and 119]

Proof. WLG, assume S is a vertical cylinder.

Suppose $R \ominus S$ is tileable. Then we can find a tiling for R as follows: In $R \ominus S$, the cut line is either covered by a domino or not. Tile all the cells in R that are also in $R \ominus S$ the same way. If there are dominoes that cross the cut in $R \ominus S$, there will be two dominoes that cross the two cuts in R (in the same row). So any row in S will either have n untiled cells, or $n-2$ untiled cells. Since this number is even, we can fill the row with horizontal dominoes. this gives us a tiling for R.

□

Unfortunately, the converse of this theorem is not true (see for example Figure 39). However, a somewhat weaker version of it is true. A deletion is called **safe** when all the cells in $R \ominus S$ have neighbors in the same directions as they had in R.

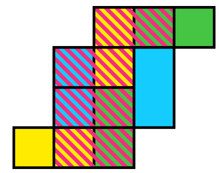


Figure 39: An example of a region R that is tileable, but $R \ominus S$ is not, for the 2-cylinder S marked pink.

Theorem 38. *Suppose S is a n-cylinder of R with n even, and that deleting it from R is safe. Then if R is tileable, then so is $S \ominus R$.*[15]

[Referenced on page 119]

[15] I suspect something stronger is true, but have not been able to work out the details.

Proof. The reduced region $R \ominus S$ has exactly the same neighbor setup as R. If R is tileable, then by the marriage theorem (Theorem 34), each set S of white cells has at least $|S|$ black neighbors. This must also hold for $R \ominus S$, since if it had a bad patch, so would R. Therefore, $R \ominus S$ is tileable. □

This method gives us a way to reduce some regions to a more manageable level.

But this method *also* gives us an algorithm to construct a tiling for R if we can find a tiling for $R \ominus S$.

Example 11. *In Figure 40 we show how a tiling can be constructed for a given region.*

First (shown in the left column of Figure 40), we successively delete 2-cylinders from the region, until the resulting region is simple enough to tile. We then tile it (if we couldn't, then its possible that no tiling exists for R. We do not know for sure, unless all the removals were safe).

Then, working backwards (shown on the right), we reinsert cylinders until we have rebuilt the original region. If the cutline goes between dominoes, we simply insert a domino perpendicular to the cutline; otherwise we insert a domino domino perpendicular to the cutline offset by a cell.

Problem 15. *Establish the tileability of the regions in 27 by deleting cylinders until each region is manageable.*

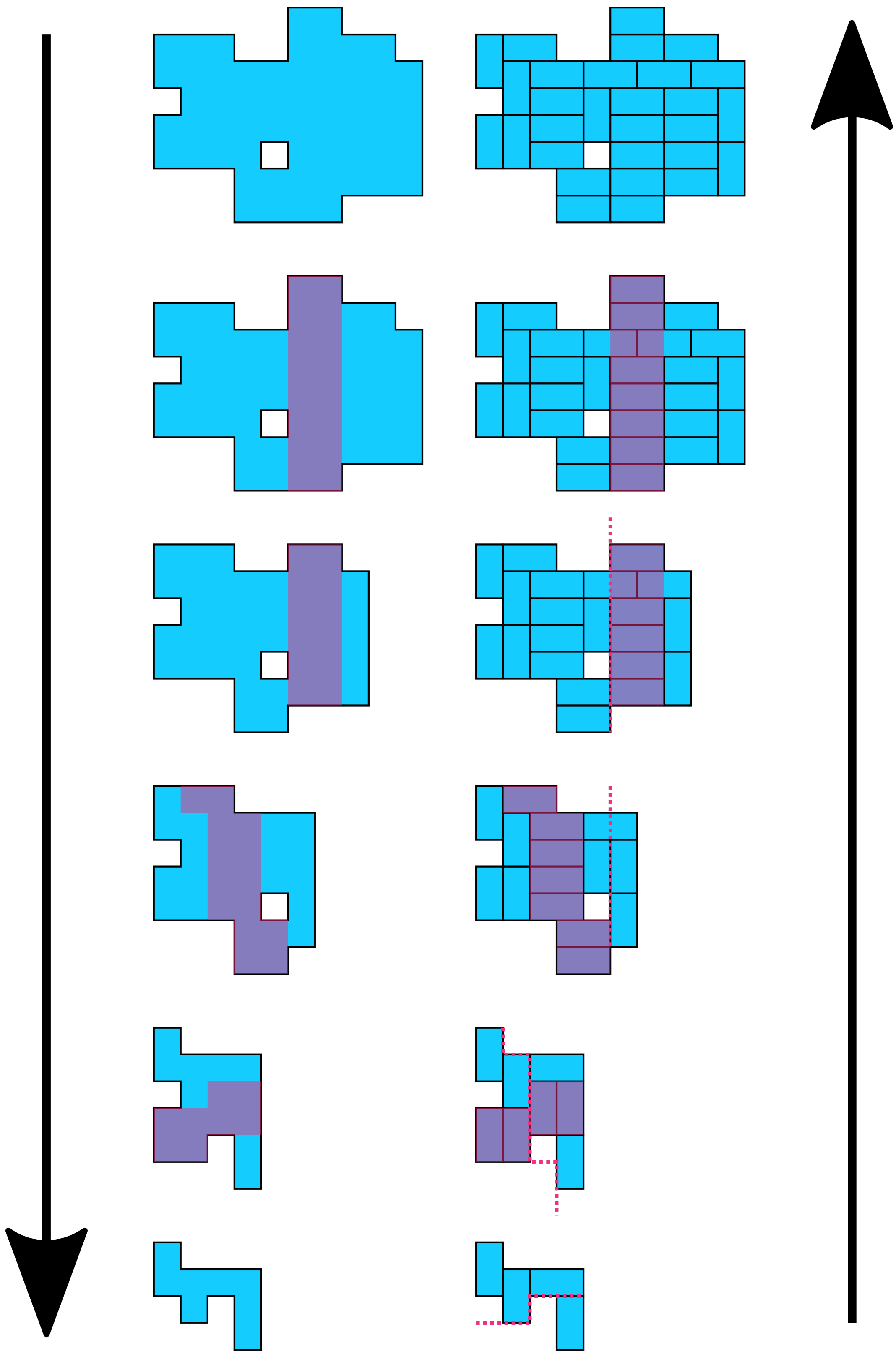


Figure 40: How to use cylinder deletion to find a tiling. On the left, from top to bottom, we delete 2-cylinders until we cannot. This region is easy to tile, shown on the bottom right. We then reinsert 2-cylinders in the reverse order (shown on the right-hand side from bottom to top). At each stage we complete the tiling in the obvious way.

Problem 16. *Give some examples of regions that do not allow us to delete a cylinder from them. Are any of them tileable?*

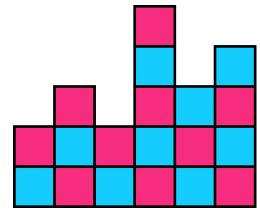


Figure 41: A bar graph with notation $B(2 \cdot 3 \cdot 2 \cdot 5 \cdot 3 \cdot 4)$.

Problem 17. *Can you characterize the regions from which we can delete cylinders?*

A **bar graph**[16] is a polyomino with columns all starting on the same horizontal line (Bousquet-Mélou et al. (1999)). A bar graph is uniquely identified by the number of cells in each column, and we write $B(a_1 \cdot a_2 \cdots a_n)$ for a bar graph with a_i cells in column i. We use exponentiation for repeated columns.

A **Young diagram**[17] is a bar graph with columns in non-increasing order (eg. Pak, 2000). In a Young diagram, we have $a_i \geq a_j$ when $i < j$. 4

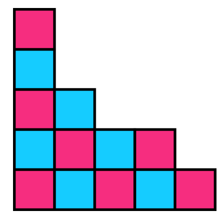


Figure 42: A Young diagram with notation $B(5 \cdot 3 \cdot 2^2 \cdot 1)$.

Theorem 39. *A Young diagram of the form $B(n \cdot n-1 \cdot n-2 \cdots 2 \cdot 1)$, with $n > 0$, is unbalanced (and therefor untileable).*[18]

[Referenced on page 45]

Proof. Apply the checkerboard coloring to the Young diagram such that the last column is black (Figure 43). Then

$$
\begin{aligned}
B - W &= \sum_{i=1}^{n} \left\lfloor \frac{i+1}{2} \right\rfloor - \sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor \\
&= \sum_{i=2}^{n+1} \left\lfloor \frac{i}{2} \right\rfloor - \sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor \\
&= \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{1}{2} \right\rfloor \\
&= \left\lfloor \frac{n+1}{2} \right\rfloor > 0.
\end{aligned}
$$

So the polyomino is unbalanced, and therefor untileable (Theorem 23). ☐

[16] Also called *Manhatten polyomino* (See for example Bodini and Lumbroso, 2009). The notation used here is from Martin (1986).

Theorem 40. *A balanced Young diagram must either have at least two adjacent columns equal, or at least two adjacent rows equal.*

[Referenced on pages 46 and 107]

Proof. If a Young diagram does not satisfy those conditions, it must be a polyomino of the form $B(n \cdot n-1 \cdot n-2 \cdots 2 \cdot 1)$, which is not balanced by Theorem 39. Therefore, a balanced polyomino cannot have this form, and so a balanced polyomino must satisfy the conditions given. ☐

Theorem 41. *A Young diagram is tileable if and only if it is balanced*[19].

[17] Also called a *Ferrers diagram* (eg. Delest and Fedou, 1993) or *trapezoidal polyomino* (eg. Bodini and Lumbroso, 2009) or *partition polyomino* (eg. Leroux et al., 1998).

[18] Compare this with Problem 10. Both are consequences of Lemma 4 in Thiant (2003). That lemma requires the notion of a *staircase* that we don't define here.

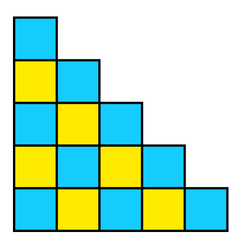


Figure 43: A Young diagram of the form $B(n \cdot n-1 \cdots 1)$, in this case $B(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$.

[19] See also Problem 13.

[Referenced on pages 46 and 119]

Proof. *If.* By Theorem 40 we know the polyomino has either two adjacent columns of equal length, or two adjacent rows of equal length. In either case, we can delete a 2-cylinder from the region. Since the resulting region must still be balanced, the conditions apply again, and we can repeat the process. The process must eventually end with the empty region, and this proves the Young diagram is tileable.

Only if. If a Young diagram is tileable it is balanced by Theorem 23. □

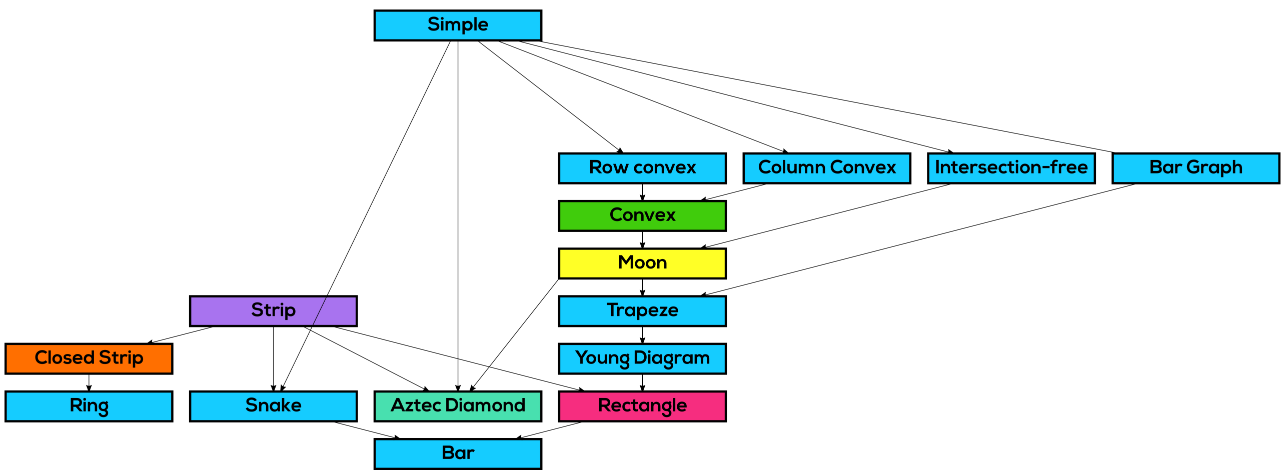


Figure 44: The relationship between some polyomino classes. Most of these are obvious.

See also Problem 33 and Theorem 53.

We already saw a class of polyominoes for which the Checkerboard Criterion (Theorem 23) is enough (namely Young diagrams, see Theorem 41), and we may wonder if there are other classes of polyominoes this is true.

There is, and this section we prove that a large class of polyominoes—that include Young diagrams—are tileable when they satisfy the checkerboard criterion. I will give three proofs of this fact, but before we get there we need a few definitions and helper theorems.

A domino in a tiling of some region is called **exposed** if at least one of its long sides lies completely on the border of the tiled region.

Theorem 42 (Neretin (2017))**.** *Any domino tiling with more than one tile has at least two exposed dominoes.*

[Referenced on pages 47, 48, 49 and 66]

Proof. We give an algorithm to produce two exposed long edges.

(1) Take any tile. If it is horizontal, look at its top (long) side. Otherwise look at the right (long) side.

(2) Introduce the coordinates: $(x, y) = (0, 0)$ at the middle of that side.

(3) If that side is free, we're done. If it is not, there must be another tile blocking it (maybe partially). Switch to that tile (or the rightmost/topmost of the two, if there are two).

(4) If it is horizontal, look at its top side. Otherwise look at the right side.

(5) Check the value of $x + y$ at the middle of that side. Make sure it increases when we step from a horizontal tile to vertical or vice versa, or (in the worst case) stays constant when we step to a tile of the same orientation.

(6) Go to step 3 and continue. It must end somewhere, for there are only so many tiles and they never repeat. (There can't be a cycle of tiles in different orientation, because $x + y$ increases when we change orientation, and never decreases. Neither can there be a cycle made of horizontal tiles only, for in that case y steadily increases in every step.)

To locate the second exposed long side, return to the initial tile and repeat everything in the opposite direction.

In fact, there must be at least two *opposite* long edges exposed.

The procedure to find the first edge always produces a right or top edge, and the procedure to find the second edge always produces a left or bottom edge. If the edges are opposite, we are done. Suppose they are not different, and WLG let the first edge be a right edge and the second edge a bottom edge. Use the procedure to find a third edge, this time going top left (rotate the coordinate system 90 degrees anticlockwise). The procedure must either produce a left or top edge; in either case, it is opposite with one of the other two exposed edges. □

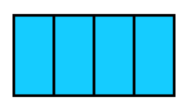


Figure 45: An example of a tiling that achieves the minimum number of exposed edges.

There are tilings that achieve the minimum number of exposed edges. An example is shown in Figure 45.

It is easy to modify this proof to apply to all rectangles (in the case of a square, all edges are "long").

Problem 18. *Extend Theorem 42 to all rectangles. Can it be extended to an even bigger class of regions?*

Rows are **comparable** if the column coordinates of one is a subset of the others.

A polyomino in a fixed orientation is **row convex** when in each row, there are no spaces between any two cells.

A **stack polyomino** is a bar graph that is row-convex and each pair of rows is comparable.[20]

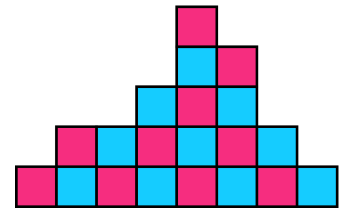


Figure 46: A stack polyomino corresponding to the vector $B(1 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 4 \cdot 2 \cdot 1)$.

[20] Also called a *trapeze* Beauquier et al. (1995).

Theorem 43. *A balanced stack polyomino contains a cylinder that can be deleted.*

[Referenced on page 49]

Proof. If the top row has two or more cells, we can remove a vertical cylinder and we are done.

Suppose then the top row has only one cell, and that it is in column k. Now we can remove a cylinder if for some $i < k$ the column differs by the next by 2 or more and for some $j \geq k$ the column differs by more than two cells.

So suppose one of these conditions fail, WLG suppose for all $i < k$ the column differs from the next by just one cell (if they are equal, we can remove a vertical cylinder).

Now divide the figure into two figures, with all m columns $i \leq k$ in S_1 and the other n columns in S_2. The deficiency in S_1 is given by $\pm \left\lceil \frac{m+1}{2} \right\rceil$. If $n < m$, it must have a abslute deficiency that is less then that of S_1. If $n > m$, there must be two columns that are equal. And if $n = m$, we have two cells in the top row. Therefor, none of the configurations are possible, and therefor both conditions must hold, and therefor, we can remove a cylinder. $\square$

Note that if we know the region is tileable, we can use Theorem 42 to prove a cylinder can be removed. However, knowing that we can remove a cylinder from a tileable region is not as useful, since we often want to remove cylinders to establish whether a region is tileable or not in the first place.

Theorem 44. *Suppose R is a stack polyomino, and S is a cylinder that can be deleted from R. Then $R \ominus S$ is a stack polyomino.*

[Referenced on page 49]

Proof. Suppose the polyomino is $B(a_1 \cdots a_m \cdots a_n)$, and $a_{m-1} < a_m \geq a_{m+1}$.

Suppose the cylinder we delete is vertical. This is equivalent to removing two adjacent columns; the resulting polyomino is a bar graph, whose columns still satisfy thee inequalities that make the polyomino a stack polyomino.

Suppose the cylinder we delete is horizontal, and it lies in columns k to k'. Then, $a_{k-1} \leq a_k + 2$, and $a_{k'} + 2 \geq a_{k'+1}$. The new polyomino is a bar graph with $B(a_1 \cdots (a_k - 2) \cdots (a_m - 2) \ldots a_{k'2} \cdots a_n)$.

Combining all the inequalities, we see this new bar graph is a stack polyomino. □

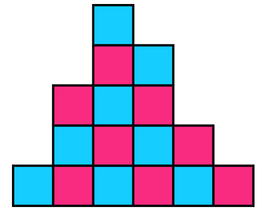


Figure 47: A balanced stack polyomino.

A **jig-saw** region is a region formed by removing all cells of one color from the top of a balanced stack polyomino (Bougé and Cosnard, 1992). A jig-saw region is sometimes, but not always, a stack polyomino. It is always a bar graph.

Theorem 45 (Bougé and Cosnard (1992))**.** *A balanced jig-saw region is tileable by dominoes.*

[Referenced on pages 49 and 119]

Proof. Suppose R is a balanced jig-saw region. Now make a partial tiling by placing a domino on each cell in the top row and its neighbor, and place horizontal dominoes on the second row starting from the outermost cells. If we remove all cells that are covered by dominoes, the resulting region R' is balanced, and a jig-saw region with one less row. Eventually, we must arrive at the empty figure, and so by the partition theorem (Theorem 2), R must be tileable. □

Theorem 46 (Bougé and Cosnard (1992), Beauquier et al. (1995))**.** *A balanced stack polyomino is tileable.*[21]

[Referenced on page 119]

[21] A fourth proof can be derrived from a more general theorem in Beauquier et al. (1995). Their result applies to tiling a stack polyomino with two bars, one horizontal and the other vertical, with no rotations allowed. If both bars have length two, we have the case of dominoes where any orientation is allowed. They introduce a lot of machinery to prove the general theorem. However, if it is made specific for dominoes, the ideas used resemble those of the second proof given here.

Proof 1. We can delete a 2-cylinder from the stack polyomino (Theorem 43) and the result a new stack polyomino (by Theorem 44) or the empty region. We can continue this process until the last region is empty. By Theorem 37 this means the region is tileable. □

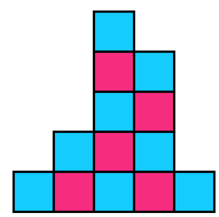


Figure 48: The jig-saw polyomino that corresponds to the stack polyomino in Figure 47.

Proof 2. (Bougé and Cosnard (1992), Korn (2004, Part of Theorem 11.1, p. 156)) There must be at least two sides with length greater than two (Theorem 42), and so at least one side that is not the base. If this side is vertical, remove the two cells connected to this side and the top corner; if this side is horizontal, remove two cells connected to this side and the left or right corner. In all cases, the remaining figure is also a stack polyomino. Eventually, we must arrive at a single domino. Re-inserting dominoes in the reversed order in the same positions now yields a tiling of the stack polyomino. □

Proof 3. (Bougé and Cosnard, 1992) Suppose the top row has an odd number of cells. Then put horizontal dominoes on the top row to leave one cell not tiled. If we remove all cells tiled, we get a new region R' which is a balanced jig-saw region, which is tileable (Theorem 45). Therefor, R must be tileable. □

3.1.4 Coloring

WE MAY WONDER if other colorings exist that could give us information when the checkerboard coloring does not.

In a sense, the checkerboard coloring gives us the best information we can hope for in the general case. However, other colorings give us information in specific cases.

We first see what happens if we add a color.

Let us color a region with three colors, amber, blue, and cherry, such that colors in each row cycle, and diagonals have the same color. We will call this type of coloring a **flag coloring**.[22]

[22] Golomb (1996, p. 4) calls this (jokingly) a *patriotic coloring*, presumably because he used the three colors of the American flag. We define generalized flag colorings in Section 5.2.

When we place a domino, it always covers two different colors, and there are three such arrangements. Let's denote the number of times a domino of each type is used by k_{AB}, k_{AC}, and k_{BC}, and the number of cells of each color in a region by A, B, and C. We have the following equations relating these values:

$$A = k_{AB} + k_{AC} \tag{3.4}$$
$$B = k_{AB} + k_{BC} \tag{3.5}$$
$$C = k_{AC} + k_{BC} \tag{3.6}$$

With a bit of algebra, we get the following solutions to the equations above:

$$k_{AB} = \frac{A + B - C}{2} \tag{3.7}$$
$$k_{AC} = \frac{A - B + C}{2} \tag{3.8}$$
$$k_{BC} = \frac{-A + B + C}{2} \tag{3.9}$$

These must all be whole numbers and non-negative (because, remember, they correspond to the numbers of dominoes), and this becomes a criterion: for a tiling to exist, it is necessary that k_{AB}, k_{AC} and k_{BC} are all non-negative integers.

Figure 49 shows an example of a region that is balanced, but does not satisfy this new criterion.

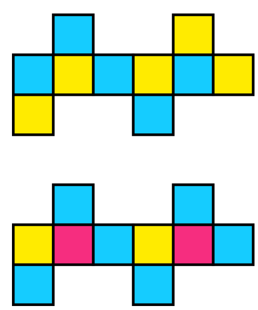


Figure 49: A region that is balanced, but does not satisfy another color criterion.

If you experiment a bit with this criterion, you will find that the regions are quite pathological and in general it is usually easy to find out they cannot be tiled *without* using the criterion. What is going on?

From the equations, we can make the following observations: k_{xy} will be an integer unless the number of cells is odd. So this part is of little help (we already know the number of cells must be even from Theorem 1). For k_{xy} to be negative, one of the colors must exceed the sum of the other two. And regions that does this tend to be spiky and uninteresting.

What if we changed the pattern? We have not used any detail of the pattern, except that neighboring cells cannot have the same color. If we color with only this restriction, it's rather easy to come up with a coloring where one color dominates. Figure 50 shows this coloring applied to Figure 27(h).

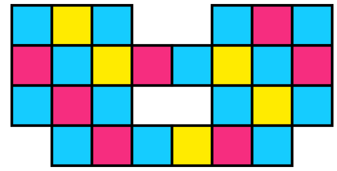


Figure 50: A coloring showing the region is not tileable.

But do we really need the two "loser" colors to be different? The answer is we don't. Let's set this up as we did before.

Color a region with amber and blue such that blue cells do not have any blue neighbors. We call such a coloring a **discriminating coloring**, with blue cells the **isolated** color. The tiles can now be of two types, blue-and-amber and amber only, and let's use k_{AB} and k_{AA} to denote how many of each tile we have.

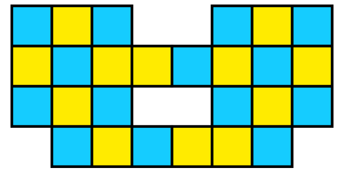


Figure 51: A coloring showing the region is not tileable. There are 14 blue cells, and 12 yellow cells. Since blue cells have no neighbors, if the region was tileable we would have at least at most as many blue cells as yellow cells.

Figure 51 shows this coloring applied to Figure 27(h).

We now have the following equations:

$$A = k_{AB} + 2k_{AA} \tag{3.10}$$

$$B = k_{AB} \tag{3.11}$$

Solving, we get

$$k_{AB} = B \tag{3.12}$$

$$k_{AA} = \frac{A - B}{2} \tag{3.13}$$

$$\tag{3.14}$$

For a tiling, k_{AB} and k_{AA} must be non-negative integers. k_{AB} will always be a non-negative integer; however, k_{AA} will be negative when $B > A$. The number k_{AA} will always be an integer as long as the number of cells is even.

If a region has a coloring for which k_{AA} is negative, we call the region **unfair** by that coloring.

Example 12. *Color the double-T as shown in Figure 12. With this coloring, $B > W$, and since no black cells have black neighbors, we conclude the tiling is impossible.*

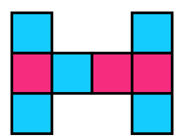


Figure 52: Double-T with alternative coloring

Problem 19. *For each remaining region in Figure 27, find a discriminating coloring by which it is unfair.*

We state this as a theorem that we will call the **two-color criterion**. Note that it actually generalizes (and contains) the checkerboard criterion. However, the concept of balanced polyominoes is still useful, and when it works it's a very efficient way to describe the particular coloring.

Theorem 47 (Two-color criterion). *If there is a discriminating coloring by which a region is unfair, then the region not tileable by dominoes.*

[Referenced on pages 52 and 119]

Proof. WLG let black be the isolated color. If a tiling exists, there must be b dominoes that cover a black and white cell, and w dominoes that cover only white cells. The number of black cells in the region is given by $B = b$, and then number of white cells $W = 2w + b$. Now if $B > W$, then $b > 2w + b$, and hence $2w < 0$, which is impossible (the number of white only dominoes must be positive or zero). Therefore, no such tiling exists. $\square$

Problem 20. *Show that we cannot get more information by adding more colors using this type of criterion.*

In a sense, the new color theorem is really just the marriage theorem in disguise. The following theorem shows the connection between them.

Theorem 48. *A region has an unfair discriminating coloring if and only if it has a bad patch.*

[Referenced on page 52]

Proof. *If.* Color the region with the checkerboard coloring. If the region has a bad patch, it has a black bad patch (Theorem 33). Let S be the biggest set that contains the black bad patch and is also a bad black patch.

Then $|\mathcal{W}(S)| < |\mathcal{B}(S)|$, and so $|\mathcal{W}(R - S)| \geq |\mathcal{B}(R - S)|$ (if it was not, we could make S bigger). Now swap all the colors in $R - S$. We now have a coloring in which there are more white cells than black cells, and all white cells only have black neighbors. Therefore we have an unfair discriminating coloring.

Only if. Suppose we have an unfair discriminating coloring of a region. Then the region is untilable (Theorem 47), and hence, in a checkerboard tiling, there is a bad patch (Theorem 34). $\square$

From this theorem, we get the following:

Theorem 49. *A region is tileable if and only if all its discriminating colorings are fair.*

[Referenced on page 119]

Proof. *If.* Suppose all the discriminating colorings of a region are fair, but it is untilable. Then, by Theorem 34 there is a bad patch, and by Theorem 48 there exists an unfair discriminating coloring, which contradicts our initial assumption, therefore, the region must be tileable. *Only if.* Suppose a region is tileable. If it had a unfair discriminating coloring, it would not be tileable (by Theorem 47), and so all discriminating colorings must be fair. □

At the beginning of this section, we said that the checkerboard coloring gives us the best information in the general case. The reason for this is that of all discriminating colorings, the checkerboard has the highest density of isolated color, and can therefor discriminate the most figures (of a certain area). I am not going to make this idea more precise here. Furthermore, it makes ideas such as flow, and (as we will see in another essay) height functions possible, which other discriminating colors do not.

3.2 *The Structure of Domino Tilings*

Some tileable regions have cells that are tiled the same way in any tiling of the region. We call such cells **frozen**.

As we will see, there are three ways in which cells can become frozen:

- When it is part of a region that can geometrically only be tiled one way (when it is part of a peak, a notion we shall define shortly).
- When other placements violate the border crossings theorem 21.
- When other placements violate the flow theorem (Theorem 25).

We will see that cells that are not frozen are always part of some closed strip, and that by rotating the closed strip we can find a new tiling of the region.

A key point from this section is the following: any two tilings are connected through a series of simple operations that involve rotations on closed strips of dominoes.

3.2.1 *Closed strips*

A **path** is a sequence of lattice points $v_1, v_2, \ldots, v_n$ such that v_{i+1} and v_i are neighbors, and no cell is repeated in the sequence.

A **strip polyomino** is a polyomino whose cells form a path. If we also have v_n neighbors v_1, then we call the strip polyomino **closed**.[23]

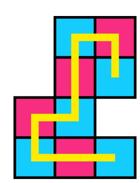


Figure 53: Strip polyomino

[23] In Beauquier et al. (1995) the word *ring* is used for a closed strip.

Theorem 50 (Beauquier et al. (1995))**.** *If we apply a checkerboard coloring to a strip polyomino, and its two ends have opposite colors, the path has an even number of cells, otherwise it has an odd number of cells.*

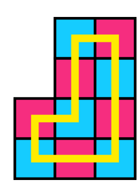


Figure 54: A closed strip

[Referenced on pages 54, 55, 62 and 101]

Proof. We note that it is true for a strip polyomino with area 1. Now suppose it holds for strip polyominoes with area k. Consider now a strip polyomino with area $k+1$, and suppose its two ends have the same color. Now remove one end, to get a path of length k. Since the new end must have opposite color of the one removed, the two ends are now of opposite color, and hence n must be even, and so $k+1$ must be odd. Suppose then the two ends have different colors. If we remove one end from the strip polyomino, the new polyomino must have ends of the same color, and hence k must be odd, and so $k+1$ must be even. We proved that the theorem also holds for $k+1$, and therefore, it must hold for all k. □

Theorem 51 (Beauquier et al. (1995)). *Strip polyominoes with even area are tileable. Closed strip polyominoes have even area, and each has eat least two different tilings.*

[Referenced on pages 54, 55, 56, 57, 59, 60, 100, 106, 117 and 119]

Proof. The cells of the strip polyomino forms a path $v_1, v_2, \ldots, v_n$, with n even. We can therefore partition the cells into sets $\{v_1, v_2\}$, $\{v_3, v_4\}$, ..., $\{v_{n-1}, v_n\}$, each containing two elements. Since in each set the two cells are neighbors (Theorem 2), the set can be tiled by a domino, and so the entire region can be tiled by dominoes .

If the strip is closed, then v_n and v_1 are neighbors, and therefor has opposite colors, and so n must be even (Theorem 50). One tiling is given above; another tiling is given by $\{v_n, v_1\}$, $\{v_2, v_3\}$, ..., $\{v_{n-2}, v_{n-1}\}$. □

When trying to find a tiling of a region by hand, it is often easier to see if you can divide it into even strips (because you can quickly draw lines through a region, and just check that endpoints have different colors). This also helps you identify suitable choke points if no tilings exist so that you can apply the flow theorem (Theorem 25) and prove it. The following two theorems illustrate this idea.

Theorem 52. *All even-area rectangles are tileable.*

[Referenced on page 166]

Proof. If the area of a rectangle is even, then either its width or height must be even. WLG say the width is even. Now partition the rectangle into rows. Each row is a strip polyomino with even area, and therefore tileable (Theorem 51), and so is the whole region (Theorem 2). □

In fact, rectangles are strip polyominoes, and what is more, rectangles with even area are closed strips.

Theorem 53. *Even-area rectangles are closed strips.*

[Referenced on page 46]

Proof. Let the width and height of the rectangle be m and n. WLG suppose the width is even. Partition the rectangle into $m+1$ strip polyominoes as follows: form the first m snakes S_i from all cells in each column except the top one; form the last snake S_{m+1} from the cells in the top rows. To each strip assign a head and tail: Except for the last snake, odd numbered snakes get a tail at the top and head at the bottom. Even-numbered snakes get a head at the bottom and a tail at the top. The last snake gets a head at the left and a tail at the right.

Now if we join the snakes head to tail in order, and join the head of the last to the tail of the first, we have a path through all the cells in the rectangle, which proves it is a closed strip. □

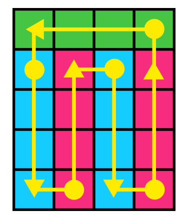


Figure 55: A rectangle is a strip polyomino.

Call a strip polyomino with odd area and with black endpoints **black**, and one with white endpoints **white**.

Theorem 54. *Let R be a white strip polyomino.*

(1) *If we remove a white cell from the strip, the remaining figure(s) are tileable.*

(2) *If we add a black cell (that is not already part of R) to the polyomino, then the resulting figure is tileable.*

The theorem holds of we interchange black and white.

[Not referenced]

Proof. If we remove a white endpoint, the resulting figure is a strip with even area and thus tileable (Theorem 51). If, on the other hand, we remove another white strip v_i, then $v_0, \cdots, v_{i-1}$ and $vi+1, \cdots, v_{n-1}$ are two strip polyominoes, both with even area. (This is because v_0 is white, and v_{i-1}, a neighbor of v_i, is black, and so Theorem 50 applies; the same for the other strip.)

If we add the black cell to one of the strip polyominoes endpoints, we have a strip polyomino with even area, and so it is tileable. If, on the other hand, we add a black cell u to a strip polyomino of odd area next to a cell that is not an endpoint, v_i (necessarily white), then we can partition the figure into three figures: $\{u, v_i\}$, $\{v_0, \cdots, v_{i-1}\}$ and $\{vi+1, \cdots, v_{n-1}\}$. The first is a domino, and hence tileable. The last two are strips with even area (as argued above) and hence are tileable. □

It is easy to extend this theorem to remove a white strip or add a black strip (that don't overlap with the original).

Let R be a region, and let R_1 be all the cells on the border of R. Let $S_i = R - R_i$, and R_i are all the cells on the border of S_i. We call R **saturnian** if for some k,

(1) $R_i = \emptyset$ for all $i > k$,

(2) R_k is tileable by dominoes

(3) R_i is a set of closed strips for all $i < k$.

S_i need not be connected. If a cell c is in R_i, we define its **level** by $\mathrm{lev}(c) = i$. (Parlier and Zappa, 2017)

The number k is called the **number of levels** of R; and the subregion R_k is called the **core**.

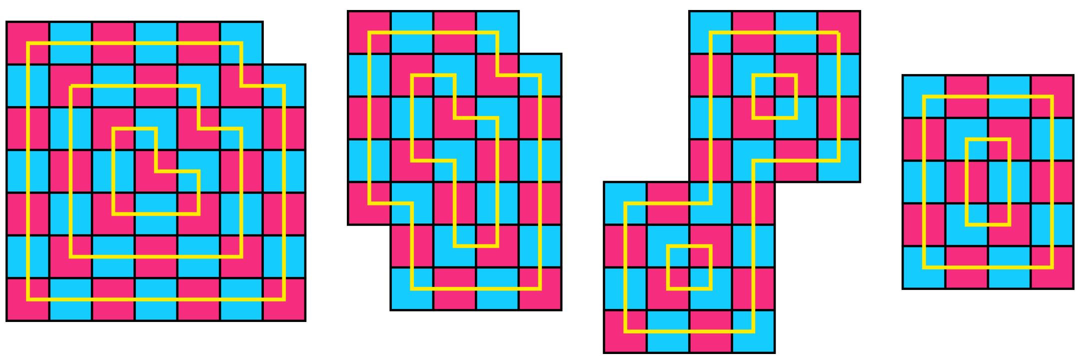


Figure 56: Examples of Saturnian regions. The third figure is an example of a Saturnian figure for which R_2 is disconnected.

Theorem 55. *Saturnian polyominoes are tilable by dominoes.*

[Referenced on page 119]

Proof. We can partition R as follows $R = R_1 + R_2 + \cdots R_k$. Each of $R_{i<k}$ is a closed strip and tileable (Theorem 51), and R_k is tileable by definition; hence, so is R (Theorem 2). □

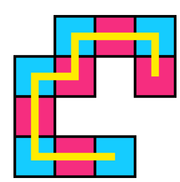


Figure 57: Snake

Problem 21.

(1) *Prove that rectangles with even area are Saturnian.*

(2) *When is a rectangle with some corners removed Saturnian?*

(3) *Prove that the core of a Saturnian region cannot contain a* 3×3 *square subregion.*

A **peak** is a cell of a region with only one neighbor in the region (Beauquier et al., 1995).

A **snake** is a polyomino that has two peaks, and every other cell has exactly two neighbors (Goupil et al., 2013). Snakes are strip polyominoes, and therefore snakes with even area are tileable.

A **ring** is a polyomino where each cell has exactly two neighbors. Rings are closed strip polyominoes.

Rings are closed strips, and therefore they have even area, and exactly two tilings.

Although we prove theorems for strip polynomials, it is not in general easy to determine whether a given polyomino is a strip polyomino (for example, it is not immediately obvious that Figure 27(e) is not a strip polyomino). On the other hand, it is easy to recognize snakes and rings.

k	$P(k)$ A000105	$S(k)$ A002013
1	1	1
2	1	1
3	2	1
4	5	2
5	12	3
6	35	7
7	108	13
8	369	31
9	1285	65
10	4655	154
α	3.6	2.3

Table 6: Number of free snakes $S(k)$ compared to the number if free polyominoes $P(k)$.

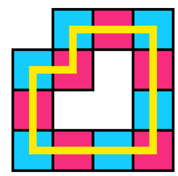


Figure 58: Ring

Problem 22. *Define the dilation of a region as the region with all cells of the original region and their neighbors. Let R' be the dilation of R. When is $R' - R$ a ring?*

Problem 23. *Define the erosion of a region as the region with all cells of the original region that has four neighbors. Let R' be the erosion of R. When is $R' - R$ a ring?*

Problem 24. *What if in the definitions in the above we use 8-neighbors instead of 4-neighbors?*

Theorem 56 (Gomery's Theorem)**.** *If we remove a white and black cell from a checkerboard colored strip polyomino, the remaining region is tileable by dominoes.*[24]

[24] The theorem, by Ralph Gomery, was originally given in only for the 8×8 square (Honsberger, 1973, p. 65–67), and the proof was slightly different. However, the main idea of the proof is the same as given here.

[Referenced on page 119]

Proof. Let the cells along the path of the strip polyomino be v_1, v_2, v_3, ..., v_k, and suppose the removed cells are v_m and v_n with $m < n$.

If the removed cells are consecutive in the path (Figure 59), then the region has a path $v_{n+1}, v_{n+2}, \dots v_k, v_1, v_2, \dots, v_{n-2}$. This is an even strip, and so is tileable (Theorem 51).

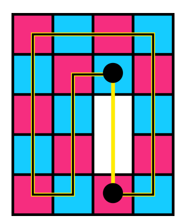


Figure 59: Removing two cells consecutive in the path.

If the removed cells are not consecutive in the path (Figure 60), they partition the strip into two strips. The ends of each strip neighbor cells of opposite color, and therefore the ends of each strip is of opposite color. Therefore, the strips are tileable, and so is the whole region. □

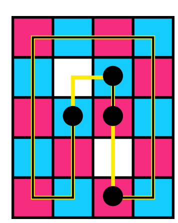


Figure 60: Removing two cells not consecutive in the path.

3.2.2 *Frozen Cells*

In Figure 62, we can see a domino can fit only one way in the two yellow cells. We can then remove these cells, and consider whether the rest of the region can be tiled. Since this is a 2×2 square, it can be, and so we conclude the whole region is tileable. The same technique works for more complicated regions, such as in Figure 61.

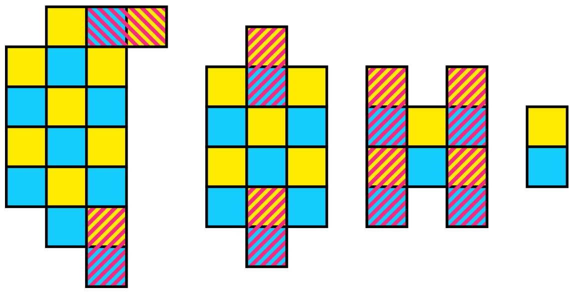


Figure 61: Illustrating peak removal.

When you apply this technique to Figure 27(b), you get a disconnected monomino after the first step, which is not tileable, and so the whole region is not tileable.

We will now formalize this process in a series of definitions and theorems.

A subregion of R is **frozen** if it can be covered by dominoes in only one way in any tiling of R.

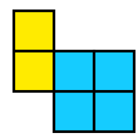


Figure 62: The yellow cells can only be covered in one way by a domino.

Theorem 57. *Suppose a region R is partitioned into two partitions, S_1 and S_2, and S_1 is tileable, and would be frozen in any tiling of R if it exists. Then the region is tileable if and only if S_2 is tileable.*

[Referenced on page 59]

Proof. *If.* If S_2 is tileable, then R is tileable by Theorem 2.

Only if. Suppose R is tileable. Because dominoes can cover the frozen partition in only one way, in a tiling of R the dominoes that are part of S_1 must also tile S_1, since it is indeed tileable. Therefore, the dominoes *not* part of S_1, must tile S_2. □

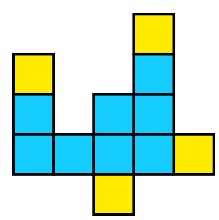


Figure 63: A region with 4 peaks, shown in yellow.

Theorem 58. *In a tileable region, a peak and its neighbor are frozen.*

[Not referenced]

Proof. There is only one way for the domino to cover the peak, and in that one way it also covers the neighbor. So if the region has a tiling, the neighbor cell can also only be covered one way. □

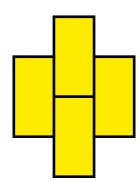


Figure 64: A region whose compact subregion is empty.

From this theorem, it follows that a region with a peak is tileable if and only if the region(s) that remain after we remove a peak and its connected cell. We can repeatedly remove peaks and their connected cells until no peaks remain. The resulting region (which may not necessarily be connected) is called the **compact subregion**. The compact subregion of R is denoted R^*. Note that R^* can be empty (Figure 64), or disconnected (Figure 65). It is possible that $R^* = R$, in which case we call R **compact**.

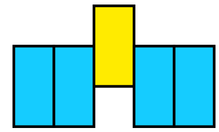


Figure 65: A region whose compact subregion (blue) is disconnected.

Problem 25. *Suppose R has no holes. Prove R^* does not have any holes.*

Problem 26. *What is the bound on $\Delta(R)$ as a function of the number of cells in R if R is compact?*

Theorem 59. *A region R is tileable if and only if its **compact subregion** R^* is tileable.*

[Not referenced]

Proof. Let $R' = R - R^*$, so that R' and R^* are partitions of R. Since all the cells in R' are frozen, by Theorem 57 R is tileable if and only if R^* is tileable. □

If a region consists out of disconnected parts, it is tileable if and only if each part is tileable (Theorem 2.) The tileability of all regions thus boils down to whether connected compact shapes are tileable.

Theorem 60 (Beauquier et al. (1995), Theorem 3.5)**.** *If a region has a unique tiling, it must have at least two peaks.*

[Referenced on pages 63, 67 and 68]

Proof. Suppose a region has no peaks, and that it has a tiling. Now construct a sequence of cells as follows: v_1 is any cell, and v_2 is the cell in the same domino. Choose v_3, a neighbor of v_2 that is not v_1 (we can do this, since v_2 is not a peak).

We can continue in this fashion, always choosing new cells that haven't been chosen before, until eventually, we are "trapped" or run out of cells. Now the last cell, v_n, must have a neighbor that is not v_{n-1}, and is already part of the sequence, say v_i. (If it did not, then v_n would be a peak). So we have a sub-sequence $v_i, v_{i+1}, \ldots v_n$ that is a closed strip. Closed strips are tileable in more than one way (Theorem 51), and therefore, the entire region must be tileable in more than one way.

Suppose the region does have one peak. Construct the same sequence as above, but choose v_1 as the peak. □

This implies that compact shapes have at least two tilings. Figure 66 shows a big figure with a unique tiling, mentioned in Pachter and Kim (1998).

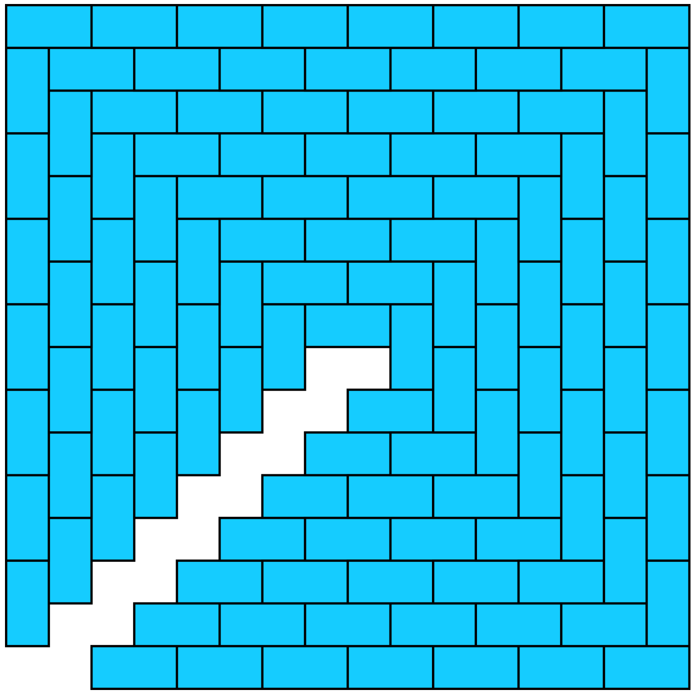


Figure 66: A figure with a unique tiling

If we recursively remove peaks and their connected cells, we are eventually left with either a compact region, or the empty region. If the former is the case, the region is uniquely tileable. Otherwise, it has more than one tiling.

3.2.3 *Tiling Transformations*

IN THIS SECTION WE LOOK AT how one tiling can be transformed into another with a series of "primitive" transformations.

Theorem 61. *A cell is not frozen if and only if it is part of a tiled closed strip.*

[Referenced on pages 62, 68 and 109]

Proof. If. Suppose a cell v_1 is part of a tiled closed strip, and its neighbor in the the same domino is v_0, and its other neighbor along the strip is v_1. Then by Theorem 51 there is another tiling of the strip, and in that tiling v_1 and v_2 are in the same domino. Therefore, there is more than one way for v_1 to be covered by dominoes, and therefore it is not frozen.

Only if. Suppose a cell v_1 can be tiled in two ways. In one tiling v_0 is the same domino, and in the other tiling v_2 is in the same domino. Then let v_3 be v_2's neighbor in the first tiling, v_4 is v_3's neighbor in the second tiling, and so on. We can always extend the sequence, but since there are only a finite number of cells in the region, eventually the next cell must be one that is already in the sequence. Moreover, this repeated must be v_0, since a cell can be in the same domino in two different tilings at most two times, and any other cells v_i is already in the same domino as either v_{i-1} and v_{i+1}.

Therefore we have a sequence of cells $v_0, v_1, \ldots, v_k$, where v_{i+1} is a neighbor of v_i, and v_n is a neighbor of v_0. Therefore, the cells $v_0, v_1, \ldots, v_n$ form a closed strip. □

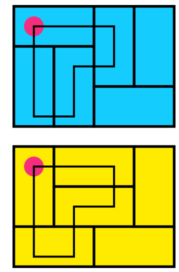
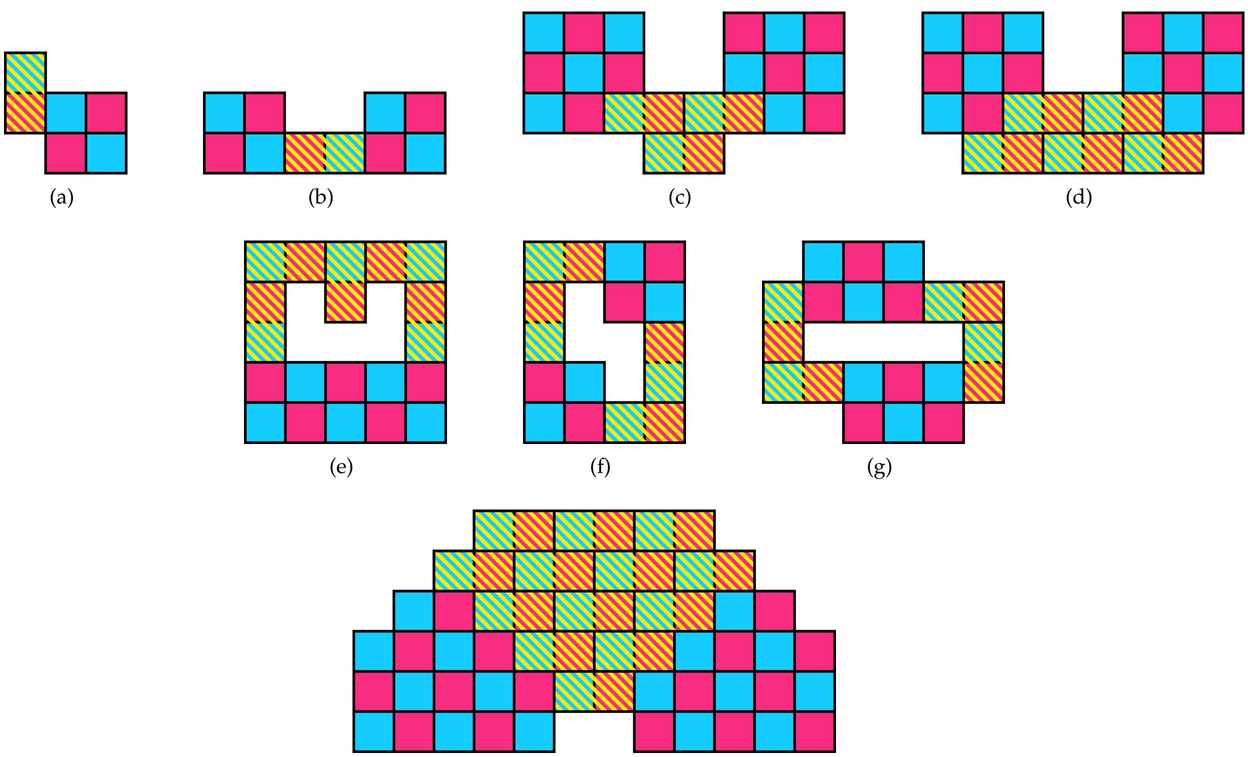


Figure 67: An example of a closed strip that contains a point and is tiled in both tilings.

The regions in Figure 68 all have frozen cells that cannot be part of any tiled closed strip.

Problem 27. *Show that if we insert a cylinder into a figure where all the cells that surround the insertion section is frozen, then frozen cells stay frozen, and the entire cylinder is also frozen.*

From the result of the problem above, we know that we can get the ratio of frozen cells to total cells as close to 1 as we want, even in figures without peaks, bridges, or holes.



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 68: Examples of regions with frozen cells, marked with yellow stripes.

Theorem 62. *If two closed strips are put side-to-side, then together they form a strip polyomino.*

[Not referenced]

Proof. Let the two closed strips be $u_1, u_2, \ldots, u_m$, and $b_1, b_2, \ldots, b_n$, and suppose u_i is neighbors with b_j. Then a strip is given by

$$u_{i+1}, u_{i+2}, \ldots u_m, u_0, u_1, \ldots, u_i, v_j, v_{j+1}, \ldots, v_n, v_0, v_1, \ldots v_{j-1}.$$

□

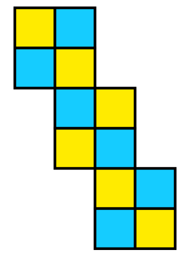


Figure 69: An example of 3 connected closed strips that do not form a strip.

Figure 69 shows an example of 3 connected closed strips (2×2 squares) that are connected, but do not form a strip.

An analogous argument shows any non-frozen cell tiled by a vertical domino, if it has a non-frozen horizontal neighbor, it can also be tiled horizontally.

In a region, if we replace a closed strip with its dual tiling, we call this operation a **strip rotation**[25] (Figure 70). When the closed strip is a 2×2 square, we call the strip rotation a **flip**, and the two dominoes that form the strip a **flippable pair**. In Theorem 61 we saw how to construct a closed strip that contains a cell from two different tilings. In the theorem below, we exploit this idea to prove that one tiling can be transform into any other tiling through a sequence of strip rotations.

[25] In Propp (2002)the other calls this operation (in the context of oriented graphs), a *face twist*.

Theorem 63 (The tiling connection theorem)**.** *We can obtain one tiling from another by a sequence of non-overlapping strip rotations.*

[Referenced on pages 67 and 110]

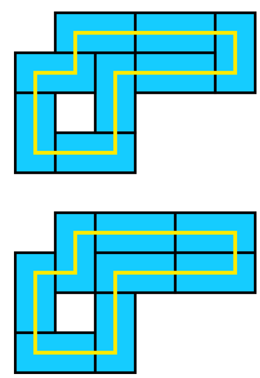


Figure 70: An example of a strip rotation.

Proof. Pick any cell with a different tiling in the two tilings. Construct the closed strip as in Theorem 61, and perform a strip rotation on that strip in the first tiling. All the dominoes in that strip now has the same tiling as in the second tiling. Repeat the process. Notice, that subsequent rounds do not change any dominoes that are already in the same position as in the second tiling, so non of the strips overlap. Since the region is finite, and we reduce the number of dominoes that differ from the second tiling in each step, the process must end when all the dominoes match. □

Theorem 64. *Let R be a tiled region, and S a subregion of R. Performing a strip rotation on R does not affect the flow on S.*[26]

[26] These ideas are more or less presented in Saldanha et al. (1995). They consider flow through cuts that do not disconnect the region.

[Not referenced]

Proof. This is clearly true when the closed strip is entirely within S or entirely out of S.

Suppose then the closed strip goes over the border of S.

If a strip outside S meets S at cells u and v, then either the strip has an even number of cells (and so u and v must have opposite colors, by Theorem 50), or an odd number of cells (and u and v must have the same colors, by Theorem 50).

In the first case, dominoes must either cross S at both points, or neither, and the net contribution of dominoes at those two points are zero. Doing a strip rotation will cause the opposite; either dominoes don't cross at either point, or they do at both. Again, the net contribution to the flow is 0.

In the second case, a domino must cross at the one point, and not the other, and therefore the contribution to the flow is $+1$ or -1, depending on the color of the cell where the domino crosses inside S. Say the crossing happens at u, and that u is black. Then the domino contributes +1 to the flow. v must be white too. When we do a rotation, there is no crossing at u, and a crossing at v. Since v is white, the contribution to the flow is 1. A similar argument shows when u is white the contribution before and after the strip rotation is -1.

So in all cases, for every strip, part of the closed strip, that meats S, we have that their contributions to the flow stays unchanged by strip rotations, and so the overall flow is unchanged. □

Theorem 65. *Every tileable region contains at least one of the following:*

- *Two peaks.*
- *A* 2×2 *square subregion.*
- *A hole.*

[Referenced on pages 64, 90, 92 and 218]

Proof. Suppose the region has a unique tiling. Then it contains two peaks by Theorem 60. Suppose then the region has more than one tiling. Pick a cell v that is not frozen. This cell must be part of a closed strip. Inside this closed strip, choose the left-most bottom-most cell. Because it is part of a closed strip, it must have at two neighbors inside the strip, and since this is a left-most bottom-most cell, it must have a top neighbor v_T and right neighbor v_R. Now consider the position u to the top of v_R. Either there is a cell, or there is not. In the former case, we have a 2×2 square. Suppose then there is not a cell in that spot. Since all the cells in the closed strip cannot lie to the left of v, they must surround u, and therefore there is a hole. □

The following theorem allows us to do induction on closed strips: it gives us a way of breaking closed strips into smaller closed strips.

Theorem 66. *Every closed strip without holes and more than 4 cells contains a smaller tiled closed strip as subregion.*

[Referenced on page 65]

Proof. By Theorem 65 a closed script must have a either a hole or a 2×2 square as subregion, and since this closed script does not have a hole, it must contain a 2×2 subregion.

Now consider how that square is tiled. Since the flux is 0, we must have that 0, 2, or 4 dominoes overlap the border.

- If 0 dominoes overlap the border, we have a flippable pair, which is a tiled closed strip, and a subregion smaller than the original.

- If 2 dominoes overlap, the cells where dominoes overlap must be neighbors. There are four ways in which we can form a closed strip with this configuration (Figure 71). In two of them the branches of the strip must overlap (Figure 71(a) and (b)), and in one of the remaining the two branches are untileable (Figure 71(c)). That leaves only one configuration (Figure 71(d)).

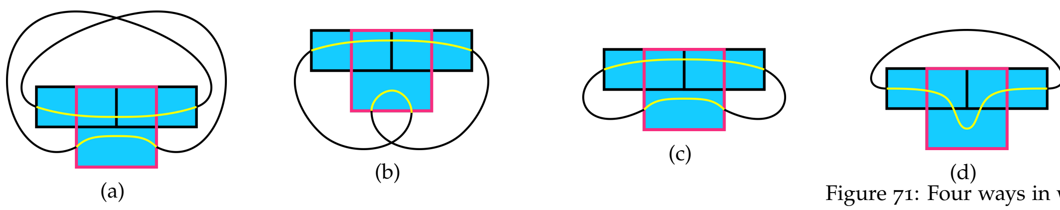


Figure 71: Four ways in which closed strips can be formed when 2 dominoes overlap.

  In this configuration we can form a smaller closed strip by removing the domino that lies completely within the square (Figure 72).

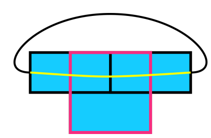


Figure 72: How a smaller closed strip can be formed this configuration.

- If 4 dominoes overlap, there are several possibilities of forming a closed strip, some are shown in Figure 73. Most of these have branches that cannot be tiled, (Figure 73(a)), or leave out one of the dominoes (Figure 73(b)); many also have branches that intersect. Two cases work (Figure 73(c) and (d)).

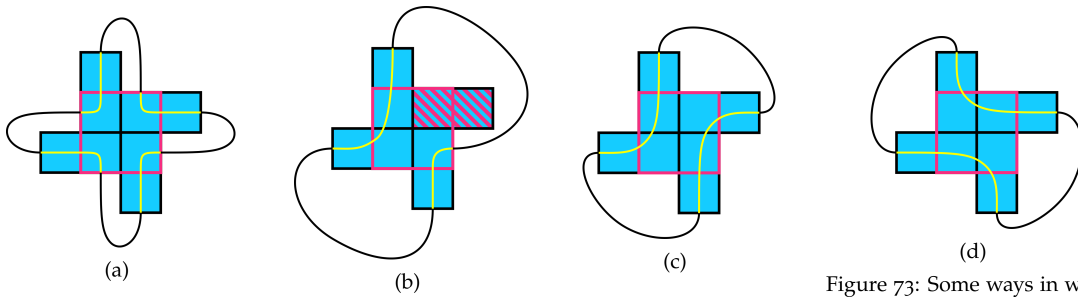


Figure 73: Some ways in which closed strips can be formed when 4 dominoes overlap.

  In these cases we can split the path into two as shown in Figure 74. Either path is shorter than the original.

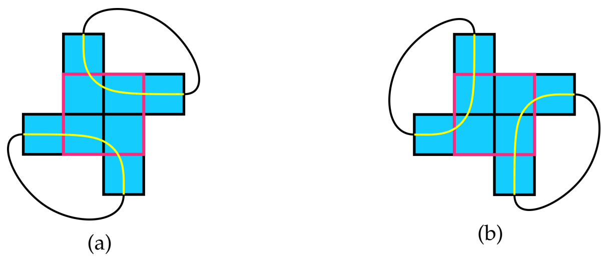


Figure 74: Splitting paths.

□

Theorem 67. *A tiling of a closed strip without holes must have at least one flippable pair.*

[Referenced on page 65]

Proof. If the closed strip is a 2×2 square, the closed strip is a flippable pair and we are done.

Otherwise, by Theorem 66, the region contains a smaller closed strip as subregion. Continue to find a smaller and smaller subregion. Since the tiling is finite, eventually we must find a 2×2 square, which is a flippable pair. □

Theorem 68. *A strip rotation (of a strip without holes) is equivalent to a sequence of flips with the same orientation.*

[Referenced on page 67]

Proof. It is true for a closed strip of 4 cells.

Find a flippable pair (it must exist by Theorem 67). If it is a 2×2 square, a flip rotates the entire strip.

Otherwise, the closed strip has more than 4 cells, and it can have either one branch (Figure 75(a)) or two branches (Figure 76(a)) to complete the strip.

(1) If it has one branch. Do the flip (Figure 75(a)). One domino is now in the right position, the other, with the branch, makes a closed strip with less than n cells, so we can do a rotation by induction. After this rotation, the complete strip is rotated.

Figure 75: A strip with one branch before and after the flip.

(2) If it has two branches. Do the flip. We can now form two closed strips with one domino in each (Figure 76(a)). Each of these can be rotated by induction. After the rotation of each, the full strip is rotated too.

Figure 76: A strip with two branches before and after the flip.

□

Theorem 69. *If a closed strip has a filled interior, a rotation of the strip is equivalent to a sequence of flips.*

[Referenced on page 67]

Proof. Find a domino with its long edge on the border of the interior (such a domino exist by Theorem 42). The long edge on the border neighbors the outer closed strip in two cells; either they lie in the same domino, or they don't.

(1) If they don't, they lie in two dominoes, who may or may not be neighbors in the strip. If they are, we can form a new closed strip by simply including the interior domino between them. Otherwise, we form two strips, a closed one with the interior domino between them, and the other what used to be between the two outer dominoes. These are closed if there is more than one domino.

(2) If they do, we flip the pair, and loop them in.

We continue this process (in each case using either of the available closed strips constructed so far) until the entire interior is enclosed within one of the closed strips $R_1, R_2, \ldots$. We may have shedded several strips $S_1, S_2, \ldots$ in the process.

Now each of R_i is a closed strip without holes, so a rotation on these are equivalent to flips. Perform the rotation on each.

Each of S_i may either be a closed strip without holes, or a single domino.

(1) In the case of the former, a rotation is equivalent to a sequence of flips, so we can do a rotation.

(2) In the case of the latter, after all the rotations, either one of the dominoes that used to be a neighbor inside the strip is now a flippable pair with it, or it is not.

   (a) If it is, do the flip.

   (b) Otherwise, perform a sequence of flips as shown in Figure 77.

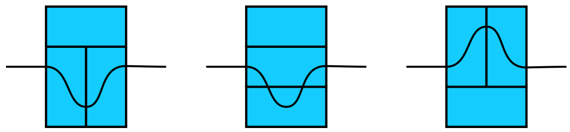


Figure 77: A sequence of flips to rotate a shedded single domino with now neighbors after the rotation.

All the dominoes in in the entire closed-strip are now rotated. Some dominoes in the interior might be in wrong positions, so they need to be restored. Pick a cell, and find a closed strip from the current and desired tiling. If this strip has an empty interior, we can perform a rotation equivalent to flips. Otherwise, we re-apply this theorem (it must end, because with each re-application we have fewer tiles and there is only a finite amount.) We repeat until the interior is completely restored.

We have performed a rotation of the outer strip, using only flips and flip-equivalent rotations. Therefore, the entire operation is flip-equivalent. □

Theorem 70 (Saldanha et al. (1995))**.** *We can obtain one tiling of a region without holes from another by a sequence of flips.*

[Referenced on page 67]

Proof. Any rotation of a strip is equivalent to flips, whether it's interior is empty (Theorem 68) or filled (Theorem 69), and any tiling can be transformed into another by a series of rotations (Theorem 63). Therefore, any tiling can be transformed into another with a series of flips.[27] □

[27] See also Rémila (2004) which gives an easier treatment of flips and domino tilings.

Theorem 71. *A tileable stack polyomino with the bottom two rows equal has at least one flippable pair.*

[Not referenced]

Proof. A stack polyomino can have at most 3 peaks (the first and last cell in the bottom row, and the top row). So if the bottom rows have the same number of cells, there can be a most one peak. But for tilings to be unique, we need at least two peaks (Theorem 60). So the tiling is not unique, and there must be another tiling. We can reach the other tiling by a sequence if flips (Theorem 70), which means the original tiling must have at least one flippable pair. □

3.3 Further Reading

In Mendelsohn (2004) the author gives a gentle introduction to the relationship between domino tilings and graph theory.

We use a variety of color arguments elsewhere in these essays (for example Sections 6.3 and 6.1.2). See Engel (1998) for other applications and ways to structure coloring arguments.

As mentioned in a side note, the marriage theorem we presented here is a specific example of the much more general theorem in the area of *matching theory*. For a survey on the development of matching history, see Plummer (1992), and for a detailed treatment, see the book Lovász and Plummer (2009). Many books on combinatorics and graph theory contain chapters on matching, see for example Harris et al. (2008), Diestel (2000) and Bondy and Murty (1976).

We showed that multiple tilings in a region is the result of certain closed strips (Theorems 60, 61). If we remove dominoes to remove all these closed strips, the tiling is unique. The number of dominoes to remove is called the *forcing number* (for example, if we remove n dominoes along the diagonal of a $2n \times 2n$ square as in Figure 66, the resulting figure has a unique tiling. These ideas are discussed in for example Pachter and Kim (1998) and Lam and Pachter (2003).

Related to the idea of frozen cells is the following: There are certain edges in a region that will never be covered by a domino in any tiling. If we consider the dual graph, the edge is called a *fixed single edge*. (In this setup, there is an edge between two cells that are neighbors. A tiling is a matching of the graph, and edges can be double or single, depending on whether there is a matching edge or not. Frozen tiles correspond to fixed double edges.) See for example Zhang (1996). The edges (in the original region) that are not crossed by dominoes in any tiling are called *fracture edges* by Fournier (1997).

Domino tilings and their statistics are of interest to physicists because they model the behavior of certain types of molecules. In this context, a domino is usually called a *dimer*, a monomino is called a *monomer*, and the tilings may be considered on more general graphs than square grids. The statistical model that deal with these tilings (or *coverings*) is called the *dimer model*. For a survey on the dimer model, see Kenyon (2000c). In the paper Cohn et al. (2001) the authors give a detailed analysis of the statistics of arbitrary regions.

For a less naive view on the topics discuss here, I recommend the following sequence of papers to get familiar with the algebra of polyomino tilings:

(1) Propp (1997) is a gentle introduction to Conway and Lagarias's work,

(2) Hitchman (2017) expands on these ideas, and includes a discussion on tiling invariants ,

(3) Conway and Lagarias (1990) is the paper where Conway and Lagarias introduces some group theoretic tools to study tilings, and

(4) Thurston (1990) is where height functions are introduced for the first time, and a polynomial time algorithm is described.

The thesis Donaldson (1996) discusses these ideas in a more leisurely fashion, and among other things, works through a proof of Thurston's algorithm.

For domino tilings, specifically, the following papers are a useful start[28]:

[28] I will give a more expansive bibliography after we covered more topics. The papers listed here deal more or less with the same topics than this essay

- *Tiling figures of the plane with two bars* (Beauquier et al., 1995).
- *Spaces of domino tilings* (Saldanha et al., 1995).
- *The lattice structure of the set of domino tilings of a polygon* (Rémila, 2004).
- *Optimal partial tiling of Manhattan polyominoes* (Bodini and Lumbroso, 2009).

4
Dominoes II

In this essay, we look further into the structure of domino tilings, how to count domino tilings, and what happens if we add or remove constraints from the tiling problem.

4.1 The Structure of Domino Tilings

We have seen that the domino tilings of a region are connected through strip rotations, and indeed that if the region is simply-connected, that it is connected through flips .

In this section we examine this structure further. We are working towards ways to answer questions like the following:

- What is the minimum number of flips we must perform on a tiling T to produce a tiling U?
- What can we say about how many (and where) flippable pairs occur in a tiling of a region?

We will do this by showing that the set of tilings of a region have a certain structure–they form a *distributive lattice*.

4.1.1 Height Functions

In this section we develop the *height function*, which will allow us to define an important tiling criterion and algorithm. We already encountered the two main ideas on which the height function is based:

(1) We can know what is the deficiency of a region by only looking at the edges on the border. (This is Theorem 27).

(2) If a region is partitioned into two subregions by a cut, the length of the cut must just about be double the absolute value of the deficiency of the one region so that enough dominoes can overlap it to compensate for the deficiency (Theorem 26).

The general idea is to look at a specific set of partitions, and see if the cut length is always long enough among these. (Instead of straight cuts, we will use cuts with a different shape.) If it is not, for one of these partitions, then we know a tiling is impossible. If it is long enough for all these partitions, then we can conclude a tiling is possible.

Suppose u and v are neighboring vertices. Then we define the **spin** $\mathrm{sp}(v, u)$ as 1 when we go from u to v and the cell to the left is black, and -1 if it is white (Rémila, 2004). Suppose $v_0, v_1, \cdots v_k$ is a path of vertices. Let us define the **height difference** of this path $h(v_0, v_1, \cdots v_k) = \sum_{i=1}^{k} \mathrm{sp}(v_k, v_{k-1})$ (Rémila, 2004)[1].

Theorem 72. *If $c_0, \cdots, c_n$ are the vertices of the border of a closed region (with $c_0 = c_n$), then if the region is balanced, $h(v_0, v_1, \cdots v_k) = 0$.*[2]

[Referenced on page 71]

Proof. Along the border of a balanced region, deficiency is 0 (Theorem 23), and the difference in white and black edges are equal to 4 times the deficiency (Theorem 27), and therefor also 0, and so $h(c_0, \cdots, c_n) = \sum_{i=1}^{k} \mathrm{sp}(c_k, c_{k-1}) = 0$. □

A path that does not cross any dominoes is called a **domino edge path** Ito (1996).

Theorem 73 (Rémila (2004), Corollary 1)**.** *Suppose we have a tiling of a region, and P and P' are two different domino edge paths from vertex u to vertex v. Then $h(P) = h(P')$.*

[Not referenced]

Proof. WLG assume the paths do not cross. (If the paths cross at a vertex w, we can consider the two paths from u to w and from w to v.)

Let $P = p_0 \cdots p_m$ and $P' = p'_0 \cdots p'_n$ (with $p_0 = p_0 = u$ and $p'_m = p'_n$). The path $Q = p_0 \cdots p_m p'_{k-1} \cdots p'_0$ does not cross any dominoes, its interior is therefore tileable, and therefor balanced (Theorem 23). So $h(Q) = 0$ (Theorem 72). But $h(Q) = h(P) - h(P')$, and so $h(P) = h(P')$. □

This shows h only depends on the start and end nodes. We now pick any vertex, v^*, and define $h(v^*) = 0$, and $h(v) = h(v^*, v_1, v_2, \cdots v)$ for any domino edge path. The function h is called a **height function** (Rémila, 2004, Definition 1).

The height function on the border of a region is the same for all tilings of a region.

[1] Ito (1996) calls this the *value* of the path, and use the symbol $\langle v_0, v_1, \cdots, v_k \rangle$ to denote it.

[2] This is a stronger version of Rémila (2004, Proposition 1).

Theorem 74 (Rémila (2004), Proposition 2)**.** *Given the height function of a tiling, we can reconstruct the tiling. Therefor, if for two tilings we have $h_T(v) = h_U(v)$ for all v, then $T = U$.*

[Referenced on pages 74 and 82]

Proof. The height difference between neighboring vertices can only be:

(1) ± 1, if there is not a domino that crosses the edge between them, or

(2) ± 3, if there is a domino that crosses the edge between them.

So for any two neighboring vertices we can determine from the height function whether a domino crosses the edge between them or not, and doing this for all neighboring pairs will show us where all the dominoes in the tiling lie. □

Theorem 75 (Rémila (2004), Proposition 3)**.** *An integer function f defined on the vertices of a region that satisfy the following properties is the height function of a tiling:*

(1) *There exists a vertex v^* such $f(v^*) = 0$.*

(2) *For neighboring vertices u and v such that $\mathrm{sp}(u, v) = 1$, we either have $f(v) = f(u) + 1$, or $f(v) = f(u) - 3$.*

(3) *If these vertices are on the boundary, then $f(v) = f(u) + 1$.*

[Referenced on page 80]

Proof. Let $w_0, w_1, w_2, w_3, w_4 = w_0$ be a path around a black cell (going anticlockwise). Then by (2) the only possibility is if there is one vertex w_j such that $f(w_{j+1}) = f(w_j) - 3$, and for other the other three vertices we have $f(w_{i+1}) = f(w_i) + 1$ (otherwise, we cannot have $f(w_0) = f(w_4)$).

A symmetric argument can be made for white cells. We can now place a domino over the edges $w_i - w_{j+1}$. Note that dominoes cannot overlap, since if they overlapped in a black cell, for example, it means there must be *two* edges $w_j - w_{j+1}$, and not just one, such that $f(w_{i+1}) = f(w_i) + 1$.

This gives us a tiling of the region. We can now verify that $f(v) = h(v)$ by induction on the distance of v from v^*, which completes the proof. □

Theorem 76. *Let R be a region and S be a strip tiled in a tiling T of R. Let U be T with S rotated clockwise. Then the height of all vertices on and outside the outside border of S stays the same. The heights of all vertices on and inside the inside border increases by 4.*

[Referenced on pages 74 and 75]

Proof. All vertices outside and on the outside border of the closed strip can be reached by paths that are unaffected by the strip rotation; so these vertices' height stays the same.

Now consider a vertex u outside the closed strip or on the outer border, and one v on the inside (or on the inside border). Before the strip rotation, there is a path from u to v from which we can calculate the height of v given that of u. This path must intersect the strip at least once, and WLG lets say it does cross it once. Since it cannot pass through a domino, it must pass between two dominoes on the closed strip. Let the vertices where this happens be w_o (on the outside border) and w_i (on the inside border).

After the strip rotation, we cannot go from w_o directly to w_i as before; instead, we need to pass around the domino. Suppose we go clockwise. Since the strip rotation was also clockwise, it means we are passing a black node on the left. We are also skipping the original section with a white domino on the left. Therefor, the net difference in height is $3 - (-1)$, which means the new height of v is now $h' = h + 4$. □

We will call a strip rotation **up** if it takes a anti-clockwise tiling to a clockwise tiling, and **down** otherwise.

A special case of this is when S is a flippable pair. An up flip increases a single vertex's height by 4. The height of a vertex between two tilings can only differ by multiples of 4, because we can get one tiling from any tiling by strip rotations[3]. (Note that while these strip rotations don't overlap, one can completely surround another, and so it is possible for the height of a vertex to differ by more than 4.)

[3] This is Rémila (2004, Lemma 1).

Problem 28.

(1) *Suppose u and v are the two vertices of a cell edge of a tileable region R. Show that for any two tilings T and U of R, that the value of $(h_T(v) - h_U(v)) - (h_T(u) - h_U(u))$ is 4, 0 or -4.*

(2) *When is it true that if we have a tiling T of a region R, and a function g defined on the vertices of R such that for any two vertices u and v that share a cell edge, we have $(h_T(v) - g(v)) - (h_T(u) - g(u))$ is 4, 0 or -4, then there is a tiling U such that $h_U = g$.*

Suppose T and U are two tilings of a region, with height functions h_T and h_U. Then we write $T \leq U$ iff $h_T(v_i) \leq h_U(v_i)$ for all i.

A relation $\leq$ defined on a set is called a partial order if it satisfies these three properties for any elements A, B, and C of the set:

$A \leq A$	Reflexive	(1)
$A \leq B$ and $B \leq A$ implies $A = B$	Commutative	(2)
$A \leq B$ and $B \leq C$ implies $A \leq C$	Anti-symmetric	(3)

Theorem 77. *The operation $\leq$ is a partial order.*

[Referenced on pages 78, 79, 80 and 81]

Proof. We need to prove the above three properties hold.

(1) $h_A(v) \leq h_A(v)$, and so $A \leq A$.

(2) $A \leq B$ implies $h_A(v) \leq h_B(v)$, and $B \leq A$ implies $h_B(v) \leq h_A(v)$, so $h_A(v) = h_B(v)$. And therefor, $A = B$ (Theorem 74).

(3) $A \leq B$ implies $h_A(v) \leq h_B(v)$ and $B \leq C$ implies $h_B(v) \leq h_C(v)$, so we have $h_A(v) \leq h_B(v)$, and so $A \leq C$.

□

Theorem 78. *Suppose that T and U are tilings that differ by a flip, and $T \leq U$. Then if X is a tiling such that $T \leq X \leq U$, then either $X = T$ or $X = U$.*

[Referenced on page 75]

Proof. From Theorem 76 it follows that for any two tilings T and U, we have $h_T(v) - h_U(v) = 4k$ for some integer k.

Now if T and U are two tilings that differ by a flip, then by Theorem 76 there is a single vertex u such that $h_T(u) + 4 = h_U(u)$, and for all other vertices v we have $h_T(v) = h_U(v)$. Suppose then there is a tiling X such that $T \leq X \leq U$. Then for all v, we must have $h_T(v) \leq h_X(v) \leq h_U(v)$. Then for $v \neq u$, we have $h_T(v) = h_X(v) = h_U(v)$, and for u we have $h_T(v) \leq h_X(v) \leq h_T(v) + 4$, or $0 \leq h_X(u) - h_T(v) \leq 4$. Since $h_X(u) - h_T(v) = 4k$ for some integer k, we must have $h_X(u) - h_T(u) = 0$ or 4, or $h_X(u) = h_T(u)$or$h_U(u)$. Therefor, for all v, we have $h_X(v) = h_T(v)$ or $h_U(v)$, and so by Theorem 74, $X = T$ or $X = U$. □

Theorem 79 (Rémila (2004), First part of Proposition 7)**.** *Suppose T and U are tilings of a region, and that $T < U$. Then we can perform an up flip on T to obtain a new tiling X such that $T < X \leq U$.*

[Referenced on pages 75 and 76]

Proof. Take the smallest vertex v such that $h_T(v) < h_U(v)$. (This implies $h_T(v) \leq h_U(v) - 4$.)

Now let v' be a neighbor of v such that $\operatorname{sp} v' = -1$. If a domino does not cross the edge $v - v'$, then by Theorem 74, $h_T(v') = h_T(v) - 1 < h_U(v) - 1 \leq h_U(v') + 1 - 1 = h_U(v')$. So $h_T(v') < h_U(v)$. But then v cannot be the smallest vertex, and so a domino *must* cross $v - v'$.

This is true for both neighbors of v with $\operatorname{sp} v' = -1$, and so we have a flippable pair. Moreover, it is anticlockwise, and so we can perform an upflip to obtain tiling X. By theorem 76 we have $h_X(v) = h_T(v) + 4 \leq h_T(v)$, and for all other vertices u we have $h_T(v) = h_X(u) \leq h_U(u)$. Thus, we have $T < X \leq U$. □

Let us write $T < U$ when $T \leq U$ and $T \neq U$. A **covering relation** $\prec$ is a such that if $T \prec U$, then $T < U$, and there is no element X such that $T < X < U$ (Davey and Priestley, 2002, 1.14, p.11).

Theorem L1. *(Davey and Priestley, 2002, 1.14, p.11) If $T \leq U$ are tilings of a simply connected region, then one of the following hold:*

(1) $T = U$.

(2) $T \prec U$.

(3) *There are tilings V_i such that $T \prec V_1 \prec V_2 \prec \cdots \prec U$.*

[Referenced on page 76]

Proof. Suppose that $T < U$. Then either there is a tiling X such that $T < X < U$, or there is not. In the latter case, $T \prec U$. In the former case, repeat the argument with T and X, and with X and U, until we recover the sequence $T \prec V_1 \prec V_2 \prec \cdots \prec U$. The process must end because no tiling is repeated (at each stage, we can write $T < X_1 < \cdots < U$ for the tilings found so far) and there are only a finite number or tilings. □

Theorem 80. *If R is a simply-connected region with tilings T and U, the following are equivalent:*

(1) $T \prec U$

(2) *T and U differ by a flippable pair, and the pair is anticlockwise in T.*

[Referenced on page 76]

Proof. Suppose $T \prec U$, which means $T < U$. Then by Theorem 79 there is a tiling X that differs by a flip from T such that $T < X \leq U$. If $X \neq U$, then we have $T < X < U$, which contradicts $T \prec U$, and so $X = U$, which means U differs from T by a single flip.

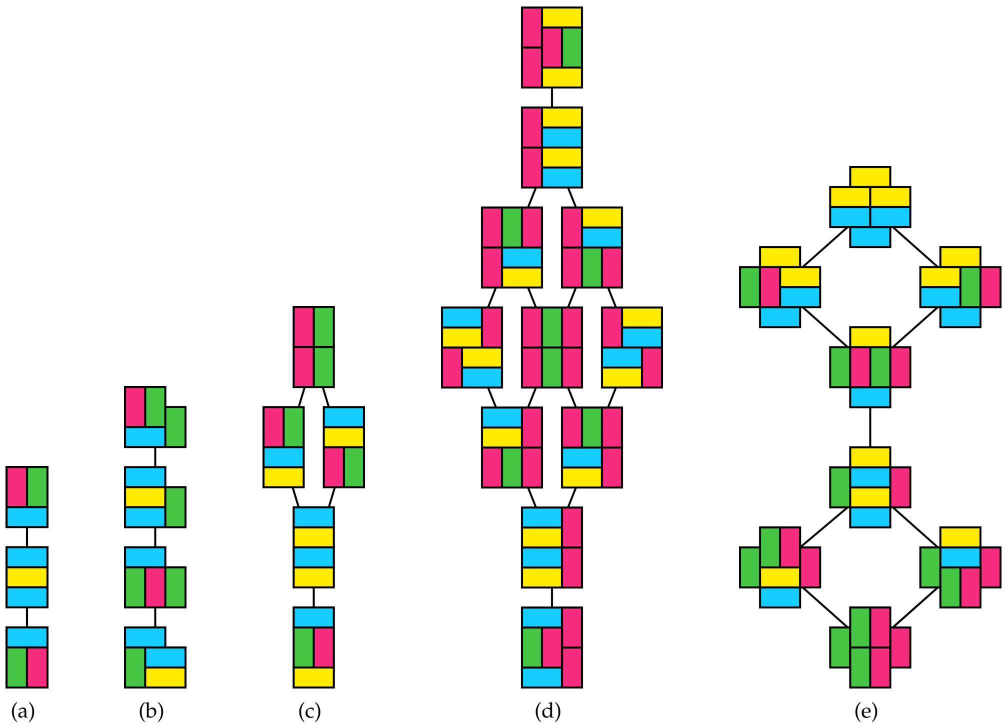


Figure 78: Examples of Hasse diagrams for the tilings of various regions without holes.

Suppose T and U differ by a flippable pair, anticlockwise in T. Now suppose X is a tiling such that $T \leq X \leq U$. Then by Theorem 78 either $X = T$ or $X = U$, and so $T \prec U$. $\square$

Alternative proof of second part. Suppose that for some V, we have $T \leq V \leq U$. This means, that $T \prec V_1 \prec \cdots \prec V_k \prec U$ (by Theorem L1), where one of V_i equals V, or, if we let $V_0 = T$ and $V_{k+1} = U$, we have $V_0 \prec V_1 \prec \cdots \prec V_k \prec V_{k+1}$. Let f_i be the flippable pair that takes V_{i-1} to V_i, and g the flippable pair that takes T to U.

Now consider the top-most (and left-most if there is more than one) flippable. The top left corner is not moved by any other flippable pair. This corner must then be different in both T and U, and so must be moved by g. And in fact, this flippable pair must equal g. Similarly, we can show bottom-most, left-most and right-most flips from f_i must all equal g. But then all f_i must equal g, but $f_i \neq f_{i+1}$ (the second one is already clockwise). So this situation is only possible if there are no V_i. Thus, no V can lie between T and U. $\square$

Theorem 81 (Rémila (2004), Proposition 7)**.** *Suppose $T < U$. Then there is a sequence of up flips that takes T to U.*

[Not referenced]

Proof. This follows from combining Theorem L1 with Theorem 80, or by induction from Theorem 79. $\square$

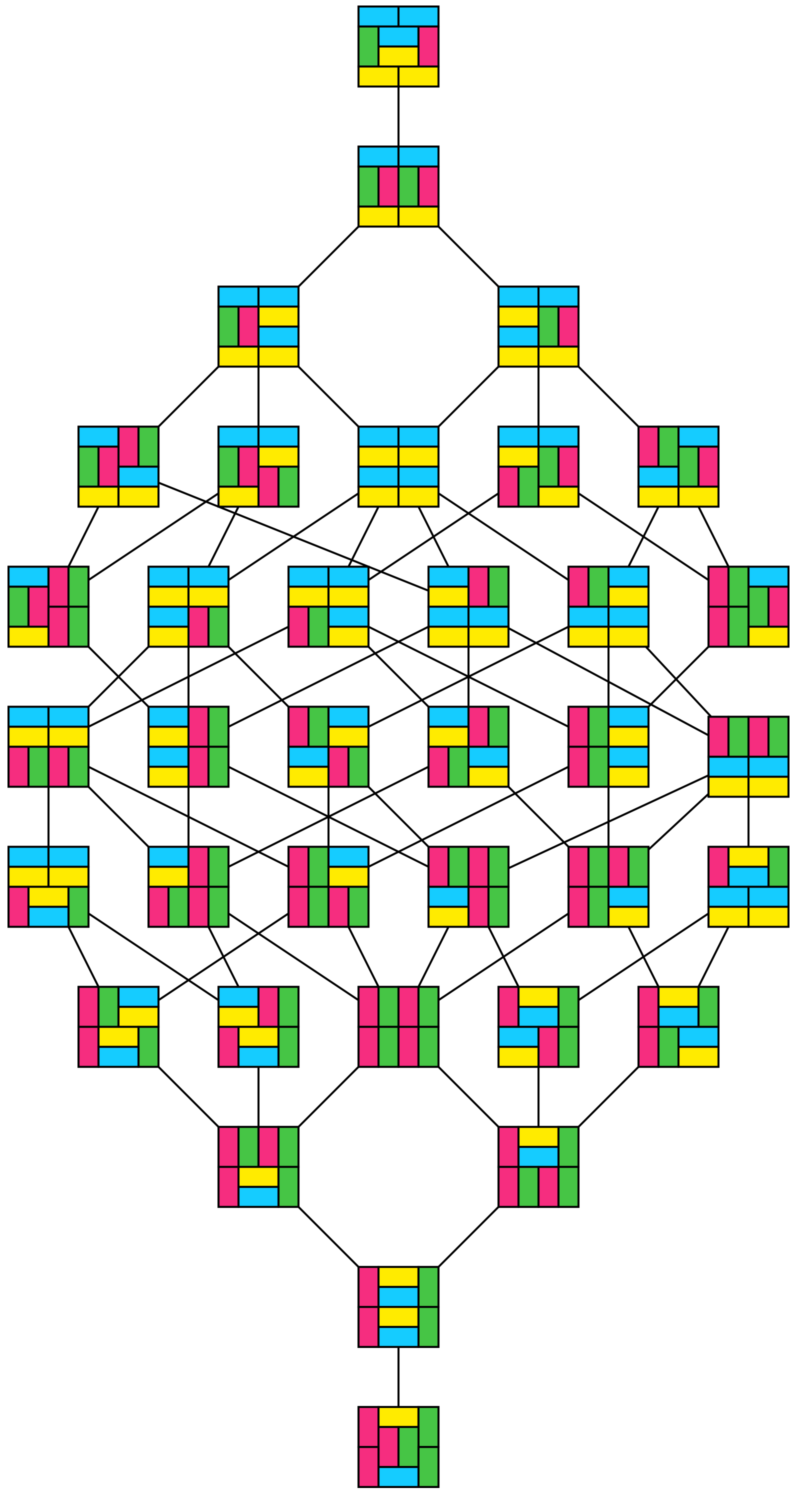


Figure 79: The flip graph of tilings of the 4×4 square.

Theorem 82. *If $A \prec T$, $B \prec T$, $A \prec U$ and $B \prec U$, then either $A = B$ or $T = U$.*

[Not referenced]

Proof. Let $A \xrightarrow{f_1} T \xrightarrow{f_2} B$ and $A \xrightarrow{g_1} U \xrightarrow{g_2} B$. If f_1 and f_2 are the same, then $A = B$. If they are different, then f_1 moves at least 2 cells that is not moved by f_2. Those same cells must be moved by g_1 (they cannot be moved by g_2 because of the orientation). But then $f_1 = g_1$, and $f_2 = g_2$, and so $T = U$. □

This theorem is generalized in Theorems L2 and L4.

Theorem 83. *Suppose that $A \leq T$, $A \leq U$, $B \leq T$, $B \leq U$, and $A \neq B$. Then there exist a tiling V such that $A \leq V$, $B \leq V$, $V \leq T$ and $V \leq U$.*

[Referenced on page 78]

Proof. If $T \leq U$, we can choose $V = T$. If $U \leq V$ we can choose $V = U$. Suppose then T and U cannot be compared.

Now A and B differ by closed strips, and say the outer ones are S_i and they surround vertices v_j. Now consider the tiling X which has all these strips in clockwise orientation. Clearly, $A \leq X$ and $B \leq X$. But clearly $X \leq T$ and $X \leq U$. □

Theorem L2. *Suppose Q is the set of all tilings bigger than both A and B. Then there exist a unique tiling V in Q smaller or equal to any element in Q.*

[Referenced on pages 78 and 79]

Proof. Let $Q = \{X_1, X_2, \cdots, X_n\}$. Now let V_1 be a tiling bigger than A and B but smaller than X_1 and X_2 (such a tiling must exist by Theorem 83). Let V_i be a tiling bigger than A and B but smaller than V_{i-1} and X_{i+1} (again, such a tiling exists by Theorem 83). Then V_{n-1} is in Q, and $V_{n-1} \leq X_i$ for all i.

To prove uniqueness, suppose there is another element V' such that it is bigger than A and B and smaller than all elements in Q. In particular, $V' \leq V$. But V is smaller than all elements of Q, and $V' \in Q$, so $V \leq V'$. Therefor, $V' = V$ (by Theorem 77.2). □

We call V the **join** of A and B and denote it by $A \vee B$. The following is basically a restatement of the definition in a more convenient form.

Theorem L3.

(1) $T \leq T \vee U$

(2) $U \leq T \vee U$

(3) *If $T \leq A$ and $U \leq A$, then $T \vee U \leq A$.*

[Referenced on pages 79 and 80]

Theorem L4. *Suppose Q is the set of all tilings smaller than both A and B. Then there exist a unique tiling V in Q bigger or equal to any element in Q.*

[Referenced on page 78]

Proof. The proof is almost exactly like the one in Theorem L2. □

We call V the **meet** of A and B and denote it by $A \wedge B$. As before, the following is basically a restatement of the definition in more convenient form.

Theorem L5.

(1) $T \wedge U \leq T$

(2) $T \wedge U \leq U$

(3) *If $A \leq T$ and $A \leq U$, then $A \leq T \wedge U$.*

[Referenced on page 79]

A set with a partial order for which the join and meet always exists is called a **lattice**.

Theorem L6. *Let T, U, and V be tilings of a region. The join and meet operations satisfy the following laws (Davey and Priestley, 2002, Thereom 2.9, p. 39):*

$$T \vee U = U \vee T \qquad T \wedge U = U \wedge T \qquad \textit{Commutative} \quad (1)$$

$$(T \vee U) \vee V = T \vee (U \vee V) \qquad (T \wedge U) \wedge V = T \wedge (U \wedge V) \qquad \textit{Associative} \quad (2)$$

$$T \vee (T \wedge U) = T \qquad T \wedge (T \vee U) = T \qquad \textit{Absorption} \quad (3)$$

$$T \vee T = T \qquad T \wedge T = T \qquad \textit{Idempotent} \quad (4)$$

[Not referenced]

Proof. Here we prove only the first of each of the four pairs of equations. The remainder is proven with dual arguments using (Theorem L5).

(1) This follows directly from the symmetry in the definition of $\vee$.

(2) Let $A = (T \vee U) \vee V$. Then, $(T \vee U) \leq A$ (Theorem L3.1), and $V \leq A$ (Theorem L3.2). From the first of these, we have $T \leq A$ and $U \leq A$. But then $U \vee V \leq A$ (Theorem L3.3), and so $T \vee (U \vee V) \leq A$ (Theorem L3.3), or $T \vee (U \vee V) \leq (T \vee U) \vee V$. Similarly, (by putting $B = T \vee (U \vee V)$) we can show $(T \vee U) \vee V \leq T \vee (U \vee V)$. Taken together, we have $T \vee (U \vee V) = (T \vee U) \vee V$ (Theorem 77.2).

(3) $T \wedge U \leq T$ and $T \leq T$, so $T \vee (T \wedge U) \leq T$. But $T \leq T \vee (T \wedge U)$, so $T \vee (T \wedge U) = T$ (Theorem 77.2).

(4) $T \leq T \vee T$ (Theorem L3.1), and (since $T \leq T$) $T \vee T \leq T$ (Theorem L3.3). Thus, $T = T \vee T$ (Theorem 77.2).

□

Theorem 84. [4] *Let T and U be tilings of a region. Then*

(1) $\min(h_T(v), h_U(v)) = h_{T \wedge U}(v)$

(2) $\max(h_T(v), h_U(v)) = h_{T \vee U}(v)$

[Referenced on page 82]

[4] This is essentially Rémila (2004, Proposition 5)

Proof. We will only prove (1), (2) can be proven symmetrically.

First, let us show that $f(v) = \min(h_T(v), h_U(v))$ corresponds to a tiling. We prove the three conditions of Theorem 75.

(1) $f(v^*) = \min(h_T(v^*), h_U(v^*)) = \min(0, 0) = 0$.

(2) Consider neighbors u and v such that $\mathrm{sp}(u, v) = 1$, and assume $h_T(u) < h_U(u)$, or $h_T(u) \leq h_U(u) - 4$. We also have $h_T(v) \leq h_T(u) + 1$, and $h_U(v) \geq h_U(u) - 3$, so $h_T(v) \leq h_T(u) + 1 \leq h_U(u) - 4 + 1 \leq h_U(v) + 3 - 4 + 1 = h_U(v)$.

    This proves if $h_T(u) < h_U(u)$, then $f(v) = h_T(v)$. But, since $f(u) = h_T(u)$, we have $f(v) - f(u) = h_T(v) - h_T(u)$, which means the second condition is satisfied. Similar arguments hold when $h_T(u) > h_U(u)$ or $h_T(u) = h_U(u)$.

(3) The above also shows the third condition is met when u and v lie on the border.

Therefor, f corresponds to a tiling.

Now let this tiling be X. Since $f = h_X = \min(h_T, h_U)$, we have $X \leq T, U$. Suppose then there is a tiling Y such that $X \leq Y \leq T, U$. Then we have $\min(h_T, h_U) = h_x \leq h_y \leq h_T, h_U$, which is possible only if $h_Y = h_X$, and therefor $Y = X$. Therefor X is the maximum tiling smaller than T and U, and therefor $X = T \wedge U$. Thus, $\min(h_T(v), h_U(v)) = h_{T \wedge U}(v)$. □

Theorem L7. *Let R be a region with a tiling. Then*

(1) *There exist a unique tiling E such that $T \leq E$ for all tilings T of R.*

(2) *There exist a unique tiling Z such that $Z \leq T$ for all tilings T of R.*

(3) *$E = Z$ if only if R has a unique tiling.*

[Referenced on pages 82 and 91]

Proof. Define $E = \bigwedge T_i$. Clearly, $T_i \leq Z$ for all i. Suppose another tiling E' has this property. We have both $E \leq E'$ and $E' \leq E$, so $E = E$ (Theorem 77.2).

Similarly, define $Z = \bigvee T_i$. $Z \leq T_i$ for all i. And if another tiling Z' has this property, we have both $Z \leq Z'$ and $Z' \leq Z$, and so $Z = Z'$ (Theorem 77.2).

Suppose R has a unique tiling T. Then clearly $T = Z = E$. Suppose on the other hand $E = Z$, and consider any tiling T of R. Then we have $Z \leq T \leq E$, or $Z \leq T \leq Z$, which implies $T = Z$ (Theorem 77.2), so all tilings of R are equal, and so R has a unique tiling. $\square$

The tiling E in the theorem above is called the **maximum tiling**, and is denoted by 1_R (this assumes the tileset $\mathcal{T}$ is understood), or if the region is understood, simply by 1. Similarly, the tiling Z is called the **minimum tiling**, and it is denoted by 0_R, or simply 0, if R is understood. The minimum and maximum tilings for some rectangles are shown in Figure 80.

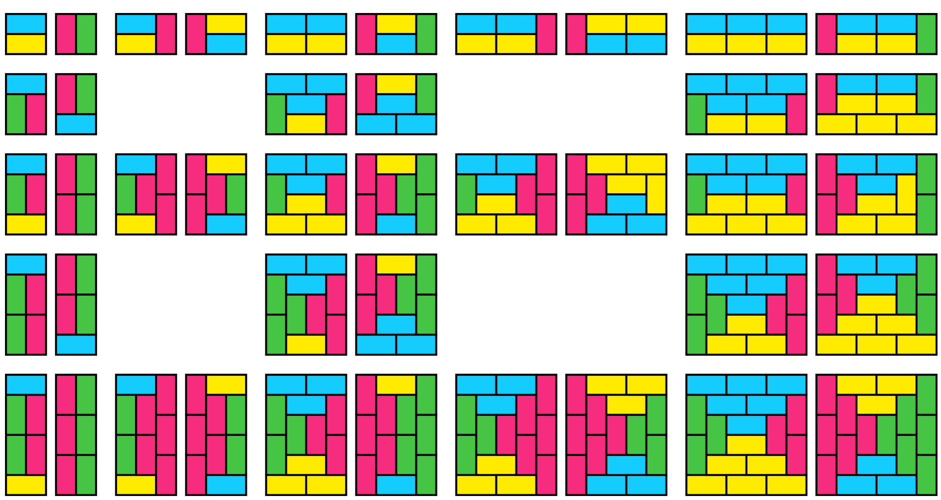


Figure 80: The minimum and maximum tilings for small rectangles.

A **distributive lattice** is a lattice that satisfies the following two distribution properties (Davey and Priestley, 2002, Def. 4.4, p. 86):

$$T \vee (U \wedge V) = (T \vee U) \wedge (T \vee V) \tag{4.1}$$

$$T \wedge (U \vee V) = (T \wedge U) \vee (T \wedge V) \tag{4.2}$$

Theorem 85 (Thurston (1990))**.** *The set of tilings of a simply-connected region forms a distributive lattice.*

[Not referenced]

Proof. We need to prove that any three tilings T, U, and V satisfy the distributive properties in Equations 4.1 and 4.2.

For the first equation, we use Theorem 84 to find an expression for the tilings on the left hand and right hand of the expression.

Let $X = T \vee (U \wedge V)$. This gives us $h_x = max(h_T, \min(h_U, h_V)) = \min(\max(h_T, h_U), \min(h_T, h_V))$.

Let $Y = (T \vee U) \wedge (T \vee V)$. Then $h_Y = \min(\max(h_T, h_U), \min(h_T, h_V))$, so $h_X = h_Y$, and so $X = Y$ by Theorem 74.

A similar argument can be used to show the second equation is also satisfied. □

Problem 29. *Find an algorithm to find the minimum and maximum tilings of a region given an arbitrary tiling of the region.*

Theorem 86. *Suppose that $T \prec X_1 \prec \cdots \prec X_m \prec U$ and $T \prec Y_1 \prec \cdots \prec Y_n \prec U$. Then $m = n$.*

[Referenced on page 82]

Proof. Let u_i be all the vertices of R. Let v_i be the vertex that changes height when we go from X_{i-1} to X_i (and let $X_0 = T$ and $X_{m+1} = U$). Similarly, let w_i be the vertices as we move along Y_i. Then we have $\sum_i h_U(u_i) - h_T(u_i) = \sum_{i=1} m + 1 v_i = \sum_{i=1} n + 1 w_i$, or $\sum_{i=1} m + 14 = \sum_{i=1} n + 14$, or $4(m+1) = 4(n+1)$, which means $m = n$. □

The **rank** $\rho(T)$ of a tiling T is defined as follows[5]:

(1) $\rho(0) = 0$.

(2) If $T \prec U$, then $\rho(T) + 1 = \rho(U)$.

Theorem L7 ensures that 0 exists, and Theorem 86 ensures this definition is consistent: it is not possible to arrive at different rank depending on the chain chosen.

Note that if $T \leq U$, then $\rho(T) \leq \rho(U)$.

Theorem 87. *Suppose that $A, B \prec X$, and $A \neq B$. Then there is a tiling Y such that $Y \prec A, B$.*

[Referenced on page 83]

[5] Rank functions can be defined for general partially orders, see for example Stanley (1986, p. 99).

Proof. Because $A, B \prec X$, and $A \neq B$, A and B differ by two flips, and that these flips do not overlap (if they did, then $A \prec X \prec B$, or $B \prec X \prec A$, which is not the case). Suppose we perform flip f from A to X, and g from X to A. We can then perform g on A to obtain Y, and f on Y to obtain B, which shows that $Y \prec A, B$. □

Theorem 88. *Let $\mu(T, U)$ be the minimum number of flips necessary to turn T into U. Then $\mu(T, U) = 2\rho(T \wedge U) - \rho(T) - \rho(U) = \rho(T) + \rho(U) - 2\rho(T \vee U)$.*

[Referenced on page 83]

Proof. Suppose a sequence of tilings from T to V via flips is $U_1, U_2, \cdots U_k$.

A triangle on A and B is a sequence of tilings such that

$A \prec X_1 \prec X_2 \prec \cdots \prec X_m \prec C$, and $B \prec Y_1 \prec Y_2 \prec \cdots \prec Y_m \prec C$.

Now there is a tiling U_i such that $U_i \prec U_{i-1}, U_{i+1}$ (unless this sequence is a triangle on T and U). By Theorem 87 there is a U_i' such that $U_{i-1}, U_{i+1} \prec U_i$. We can repeat this procedure until we have a triangle. Note that the operation does not change the number of flips.

Now let C be the top of this triangle. Then $T, U \leq C$, thus $T \vee U \leq C$. And since $T, U \leq T \vee U$, by Theorem we have a flip path from T to U via $T \vee U$, and this path of length $2\rho(T \wedge U) - \rho(T) - \rho(U)$. And since $T \vee U \leq C$, this path must be smaller than the path through C, and hence, the shortest path via flips between two tilings is through the join, given by $2\rho(T \wedge U) - \rho(T) - \rho(U)$.

A symmetric argument shows another shortest path is through the meet, which gives us the length $\rho(T) + \rho(U) - 2\rho(T \vee U)$.

So $\mu(T, U) = 2\rho(T \wedge U) - \rho(T) - \rho(U) = \rho(T) + \rho(U) - 2\rho(T \vee U)$. □

Theorem L8 (Parlier and Zappa (2017))**.** *Let T and U be two tilings of a region. Then $\mu(T, U) \leq \rho(1)$.*

[Not referenced]

Proof. Suppose that $\rho(T) + R(U) > \rho(1)$. By Theorem 88 $\mu(T, U) = 2\rho(1) - (\rho(T) + \rho(U))$, so $\mu(T, U) < \rho(1)$. Suppose on the other hand that $\rho(T) + \rho(U) \leq \rho(1)$. By Theorem 88, $\mu(T, U) = \rho(T) + \rho(U) - 2\rho(T \vee U) \leq \rho(T) + \rho(U) \leq \rho(1)$. □

We next give the maximum ranking for rectangles and Aztec diamonds. Using the above theorem, we know what is the most flips we need to turn one tiling of one of these figures into another tiling.

Theorem 89 (Parlier and Zappa (2017)). *For rectangles,*

$$\rho(1_{R(m,n)}) = \begin{cases} \frac{mn^2}{4} - \frac{n^3}{12} - \frac{n}{6} & \text{for } n \text{ even} \\ \frac{mn^2}{4} - \frac{n^3}{12} + \frac{n}{12} - \frac{m}{4} & \text{otherwise} \end{cases}. \tag{4.3}$$

For squares,

$$\rho(1_{R(n,n)}) = \frac{n^3 - n}{6}. \tag{4.4}$$

[Not referenced]

Table 7 shows some values for $\rho(1_{R(n,n)})$.

	2	4	6	8	10	12
1	0	0	0	0	0	0
2	1	3	5	7	9	11
3	2	6	10	14	18	22
4	3	10	18	26	34	42
5	4	14	26	38	50	62
6	5	18	35	53	71	89
7	6	22	44	68	92	116
8	7	26	53	84	116	148
9	8	30	62	100	140	180
10	9	34	71	116	165	215
11	10	38	80	132	190	250
12	11	42	89	148	215	286

Table 7: $\rho(1)$ for $R(m,n)$.

Theorem 90 (Parlier and Zappa (2017)). *For Aztec diamonds,*

$$\rho(1_{A(n)}) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

[Not referenced]

Table 8 shows some values for $\rho(1_{A(n)})$.

n	A000330
1	1
2	5
3	14
4	30
5	55
6	91
7	140
8	204
9	285
10	385
11	506
12	650

Table 8: $\rho(1)$ for $A(n)$.

The following relates the height function of a tiling with its ranking.

Theorem 91. *The ranking of a tiling is given by*

$$\rho(T) = \frac{1}{4}\left(\sum_i h_T(v_i) - \sum_i h_0(v_i)\right).$$

[Not referenced]

Proof. Clearly, $\rho(0) = 0$. Suppose now the above is true for all tilings T with $\rho(T) < k$. Suppose now T is any tiling with rank $\rho(U) = \rho(T) + 1$. This means, that U and T differ by a flip, and so there is a vertex v_m such such $h_U(v_m) = h_T(v_m) + 4$, and $h_U(v) = h_T(v)$ for all other vertices. Thus

$$\begin{aligned} \rho(U) &= \rho(T) + 1 \\ &= \frac{1}{4}\left(\sum_i h_T(v_i) - \sum_i h_0(v_i)\right) + 1 \\ &= \frac{1}{4}\left(-4 + \sum_i h_U(v_i) - \sum_i h_0(v_i)\right) + 1 \\ &= \frac{1}{4}\left(\sum_i h_U(v_i) - \sum_i h_0(v_i)\right). \end{aligned}$$

□

A **local minimum** is a vertex v with neighbors v', such that $h(v) < h(v')$ for all neighbors v'. Similarly, a **local maximum** is a vertex v such that $h(v) > h(v')$ for all its neighbors v'.

Theorem 92 (Thurston (1990))**.** *Algorithm for finding a tiling of a region if it exists.*

[Referenced on page 119]

Proof.

(1) An interior vertex is a local minimum or maximum if and only if it lies at the center of a flippable pair.

(2) The minimal tiling of a figure has no local maximums in the interior.

(3) If it did, we could get a tiling smaller than the minimum by performing a flip.

(4) Therefor, the vertices for which $h_0(v)$ is maximum lie on the border. Let v be such a vertex, and let u and w be its neighbors so that u, v and w is in clockwise order. Note that $h_0(u), h_0(w) < h(v)$, so the edge between u and v must be white, and the edge between v and w must be black.

(5) Therefor v cannot be a corner, which means u, v and w lie in a straight line.

(6) There cannot be a domino path from v to the interior neighboring vertex v', otherwise it will be higher than the maximum (since it has a black sell on the left). Therefor, a domino must cover the edge between v and v'. We found one domino of the tiling 0 (if it exists).

(7) We can now remove the two covered cells, update the height function on their edges, and repeat the process. If, when we update the height function we cannot assign consistent heights, a tiling is impossible.

□

A **direct path** from one vertex to another is a path from the one vertex to the other so that it always has a black cell on the left. Note that this path is not symmetric (See Figure 81). The **distance** between two vertices is the length of the shortest direct path between them (note that since the path has black cells on the left, the spin from one vertex on the path to the next is always 1).

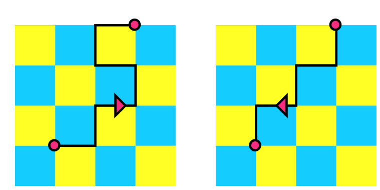


Figure 81: A direct path between two nodes. If we change the direction of the path, some edges are different.

Removing a cylinder from a region does not affect the heights of the vertices of the border that remain. To make this statement precise, we need a function that maps the cells of the reduced region to the cells of the original region. If R is a region, and S is a 2-cylinder, then $f_S : R \ominus S \to R$ maps the cells in $R \ominus S$ to their original counterparts in S.

Theorem 93. *If R is a region and S is a removable 2-cylinder, and h a height function. Then $h_{R \ominus S}(v_i) = h_R(f_S(v_i))$ for all vertices.*

[Not referenced]

Proof. Let v_k be a vertex on the border of $R - S$, with height function defined relative to the vertex v_0. Let $v_0, v_1, v_2, \cdots v_k$ be a path that does not cross any dominoes. The cylinder can cut through this path zero times, once, or twice.

(1) If it does not cut through the path, then the height of v_k is the same in both R and $R \ominus S$.

(2) If it cuts through the path once, there is one extra white and one extra black edge in R, which means the net effect is 0.

(3) If it cuts through the path twice, there is two extra white and two extra black edges, and the net effect on the height is 0.

In all cases, $h_{R \ominus S}(v_k) = h_R(f_S(v_k))$ □

Theorem 94. *Let S be a removable cylinder of R, and suppose u and v are vertices with not in the interior of the cylinder, or the borders of the cylinder shared with R. Then*

$$d_{R \ominus S}(u, v) < d_R(f(u), f(v)) < d_{R \ominus S}(u, v) + 4.$$

[Not referenced]

Proof. If u and v are on the same side of the cylinder, the shortest path between them is unaffected, and therefor the length is the same.

If they lie on opposite sides of the cylinder, we can find a shortest path between them that has at most 4 edges in the cylinder. This is because a piece of the path with more than 4 edges must have a section of 3 consecutive edges where the path goes around a cell, and can be changed to go around a different cell outside the cylinder. See Figure 82 for an example. And when we remove the cylinder, this path is at most 4 units shorter.

□

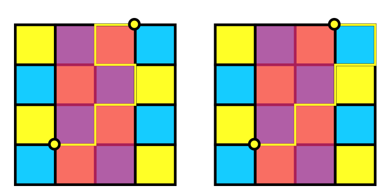


Figure 82: Rerouting a direct path.

Problem 30.

(1) *The two theorems above show us that if* $|h_{R \ominus S}(f(x), f(y))| \leq d_{R \ominus S}(f(x), f(y))$, *then* $|h_R(x,y)| \leq d_R(x,y)$. *Use this to show that if* $R \ominus S$ *is tileable, then so is* R. *You also need to take the vertices of* S *into account. This is a different way to prove Theorem 37.*

(2) *Show that the converse does not hold by checking that Figure 39 provides a counter example.*

Theorem 95. *A simply-connected region* R *is tileable by dominoes if and only if* $\Delta(R) = 0$ *and* $|h(x,y)| \leq d(x,y)$ *for all vertices* x *and* y *on the border.*

[Referenced on page 119]

Proof. Suppose that there are nodes x and y such that $h > d$. Let S be a subregion of R between a shortest direct path between x and y and the border of R.

If R is tileable, then $\phi(S) = \Delta(S) = \frac{h+d}{4}$. But the only place where dominoes can cross is on the direct path between x and y. Note that because of the paths shape, we have

$$\frac{d}{3} \leq \phi(S) \leq \frac{d}{2}. \tag{4.5}$$

Thus $\frac{h+d}{4} \leq \frac{d}{2}$, or $h + d \leq 2d$, or $h \leq d$, which contradicts our assumption, and therefor R is not tileable.

On the other hand, suppose R is not tileable. Then there is a bad patch S. This bad patch cannot lie completely in the interior of the region (because all such cells have 4 neighbors in R, and therefor cannot be a bad patch). So suppose the bad patch shares a border with R between vertices x and y. WLG assume the patch is black (if it is white, we can also find a black bad patch by Theorem 33). Now take a shortest direct path between x and y, and let S' be the subregion between this path and the border of R, with $\phi(S') > \Delta(S')$ (This is possible since the region is not tileable, and either the flux must be bigger or smaller; since there are two possible subregion, we can find one with the necessary constraint.)

By Equation 4.5 we have $\frac{h+d}{4} < \phi(S') \leq \frac{h}{2}$, so $h + d < 2h$, or $d < h$. □

The section *Further Reading* gives some references in dealing with regions with holes, which we don't cover here.

4.1.2 *Forced flippable pairs*

The following section is from Kranakis (1996). A **pyramid**[6] is stack polyomino with the following forms:

[6] This is a slightly different definition than given in Kranakis (1996), which does not account correctly for all the cases in the pyramid lemma.

- For an even number of $2k$ columns: $B(3 \cdot 4 \cdot 5, \cdots (k+2)^2 \cdots 5 \cdot 4 \cdot 3)$. See Figure 83.
- For an odd number of $2k+1$ columns: $B(3 \cdot 4 \cdot 5, \cdots (k+2)^3 \cdots 5 \cdot 4 \cdot 3)$. See Figure 84.

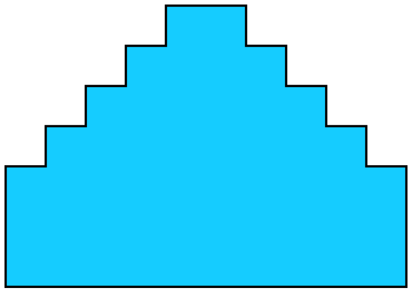


Figure 83: An even pyramid with vector $B(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7^2 \cdot 6 \cdot 5 \cdot 4 \cdot 3)$.

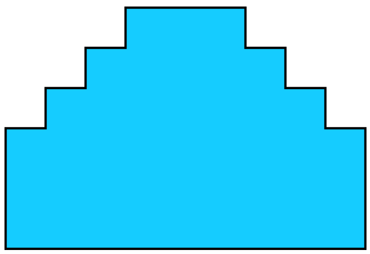


Figure 84: An odd pyramid with vector $B(3 \cdot 4 \cdot 5 \cdot 6^3 \cdot 5 \cdot 4 \cdot 3)$.

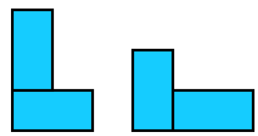


Figure 85: An L-configuration.

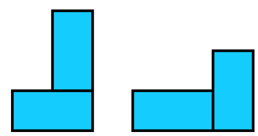


Figure 86: An R-configuration.

An **L-configuation** is two dominoes at 90 degrees that form a "left corner", (Figure 85); an **R-configuration** is two dominoes at 90 degrees that form a "right corner" (Figure 86).

If each of these occur on the same height within a tiling with the L-configuration on the left, we call the combination an **LR-configuration**.

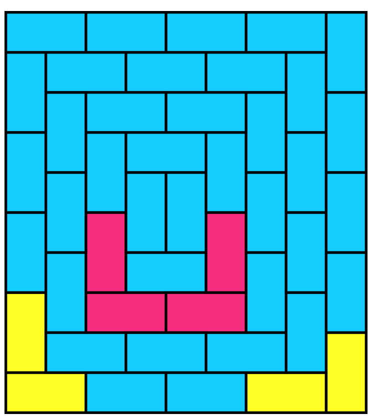


Figure 87: This tiling has several LR-configurations. One is shown in yellow, another one in red.

Theorem 96 (The Pyramid Lemma, Kranakis (1996), Lemma 4)**.** *If a domino tiling of a rectangle is given, and a LR-domino configuration is* *present in the tiling, then there is a flippable pair present in the smallest pyramid that contains the LR-configuration on its base.*

[Referenced on pages 89, 90, 91, 92, 94, 123 and 219]

Proof. There are a few cases to consider, but the general pattern is the same. Take the LR configuration shown. The left horizontal domino can either be covered by another horizontal domino (in which case we have a flippable pair), or by a vertical domino. This vertical domino either has another vertical domino to the right (in which case we have a flippable pair) or a horizontal domino (forming an L-configuration). A similar argument on the right shows we either have a flippable pair, or another R-configuration. We continue this process, until eventually a flippable pair is forced.

Figures 88 and 89 shows all the different cases for different LR-configurations and whether the smallest pyramid is even or odd. □

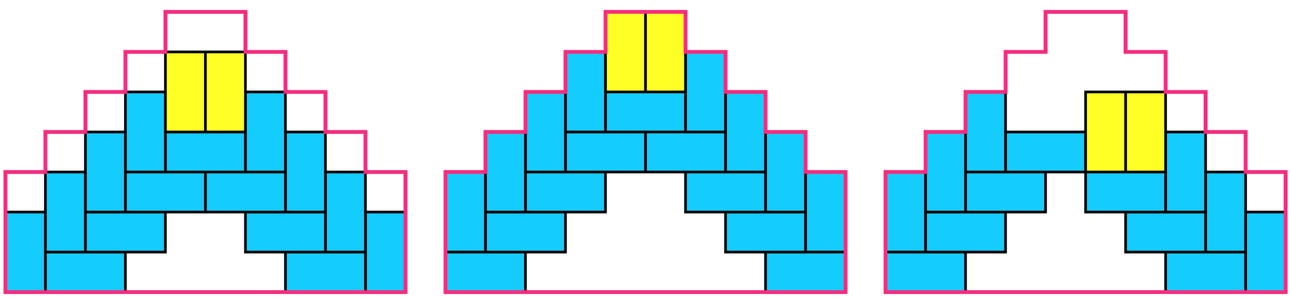


Figure 88: Forced flippable pair in even pyramids.

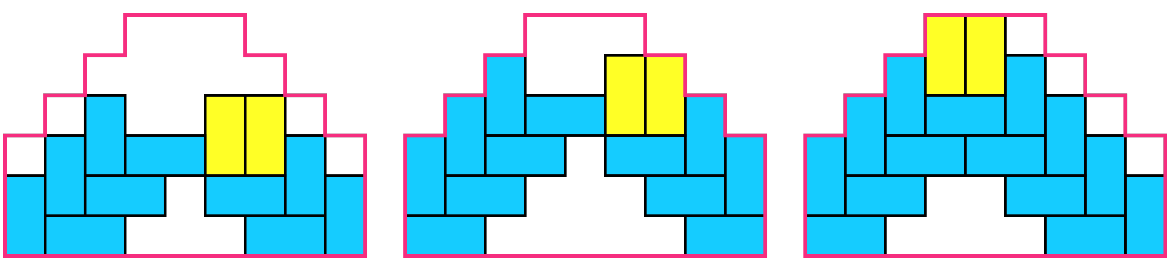


Figure 89: Forced flippable pair in odd pyramids.

Theorem 97. *Suppose there is a vertical domino on the bottom border or a rectangular tiling, with no other vertical dominoes between it and the left (or right) border of the rectangle. Then there is a flippable pair inside the pyramid with the vertical domino in one corner, and the other corner just outside the left (or right) rectangle border.*

[Referenced on pages 92, 93, 94 and 123]

Proof. The logic is exactly the same as used in the proof of Theorem 96: If we try to fill the tiling without adding any flippable pairs, we are eventually forced to add a flippable pair. Figure 90 shows the setup.

□

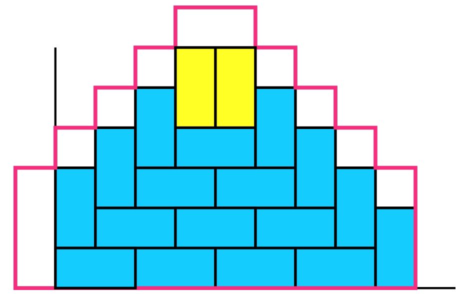


Figure 90: A forced flippable pair in the pyramid between a vertical domino and the rectangle border.

Theorem 98. *If there is an R-configuration in a tiling if a rectangle, and we draw a diagonal to the top left, there is a flippable pair that crosses this diagonal.*

[Referenced on page 123]

Proof. Again, the logic is the same as in Theorem 96. If we fill in the tiling by avoiding flippable pairs, we are eventually forced to add a flippable pair. Figure 91 shows the setup.

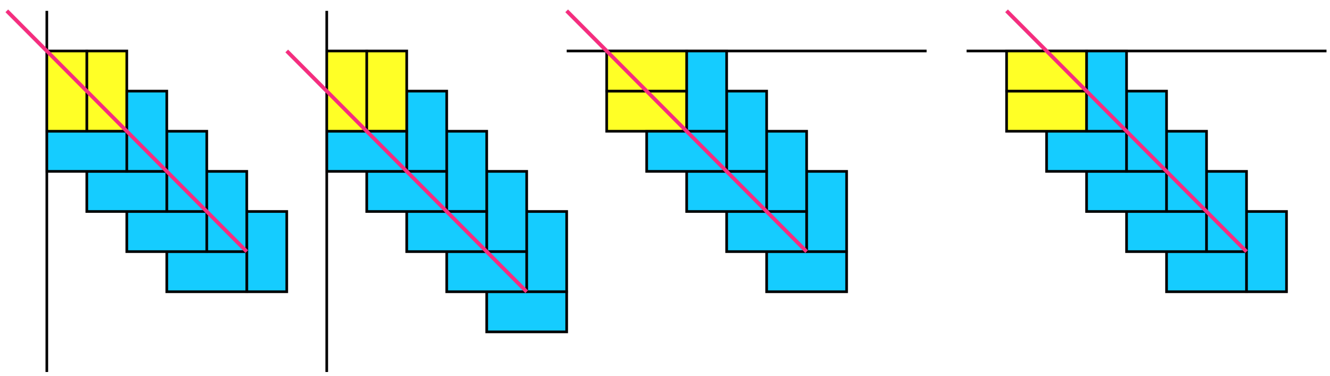


Figure 91: Forced flippable pair occurs on the diagonal that starts from the R-configuration.

□

Theorem 99 (Kranakis (1996), Theorem 5)**.** *Domino tilings of rectangles with both sides of length greater than 1 must always have a flippable pair.*

[Referenced on page 94]

Proof. This follows directly from Theorem 65, since if the rectangle has all sides greater than 1, it has no peaks, and also it has no holes.

We give here an alternative proof given in Kranakis (1996).

Suppose the region rectangle is $R(m, n)$. We prove it by induction on the area mn.

If $m = 2$, then either the first two cells are covered by two horizontal dominoes (a flippable pair), or by a vertical domino as in Figure 101. In the latter case, the next two cells are either covered by two horizontal dominoes (forming a flippable pair), or a vertical domino, forming a flippable pair with the first vertical domino. In all cases we have a flippable pair, and by symmetry the same is true if $n = 2$.

Suppose then both $m, n > 2$. We know (Theorem 1) that mn is even, so either m or n must be even. WLG suppose m (the horizontal dimension) is even. If there are now vertical dominoes touching a horizontal border, we can strip off a row to get a new rectangle $R(m, n-1)$ with area $m(n-1) < mn$, and the theorem follows by induction.

Suppose there are vertical dominoes along the (say bottom) border. Since m is even, there must be at least two (Theorem 21). Choose two adjacent such dominoes. Then all the dominoes next to the border are horizontal, and by Theorem 96 there is a flippable pair in the pyramid induced by the corner configurations. □

Problem 31. *Show that if a tiling of $R(m, n)$ with either $m > 3$ or $n > 3$ has two overlapping flippable pairs in a subregion S, then it has two flippable pairs that don't overlap.*

Theorem 100.

(1) *In a rectangle, there are at most two tilings with exactly only one flippable pair.*

(2) *If there are such tilings, the one is a rotation of the other.*

(3) *$R(m, n)$ have 0 such tilings if $m > n + 2$ or $n > m + 2$.*

(4) *$R(m, n)$ has 1 such tiling if $m = n + 2$ or $n = m + 2$.*

(5) *$R(m, n)$ has 2 such tilings otherwise, that is $m < n + 2$ and $n < m + 2$, or $n - 2 < m < n + 2$.*

[Referenced on page 92]

Proof. A tiling with only one flippable pair must be either minimal or maximal (since its rank can only be reduced or increased by flipping the flippable pair, it cannot be both increased and decreased since there is only one flippable pair.) Since there is exactly one minimal and maximal tiling (Theorem L7), there can be at most two tilings with flippable pairs.

Suppose there are two tilings with exactly one flippable pair.

(1) If $m = n$, we have a square. Now if we rotate one tiling by 90°, we get a different tiling with one flippable pair. But there can be at most two, so this tiling must equal the other tiling, and so the two tilings are rotations of each other.

(2) If $m \neq n$, suppose one tiling is not symmetric w.r.t. a 180° turn. Then by turning it, we can get another tiling. Again, there must be at least two, so it must equal the other tiling.

> If both are symmetric with respect to a 180° turn, then they must be equal (?), but then there cannot be two tilings with one flippable pair.

So in all cases, there if there are two tilings with one flippable pair, the two tilings are rotational copies of each other.

$R(2, m)$ has more than two flippable pairs for $m > 4$. Each vertical domino must be adjacent to another vertical domino, forming a flippable pair. But if there is only one such, all other dominoes must be horizontal, and so if $m > 4$, there must be at least 3, and they and any arrangement must result in at least one more flippable pair.

$R(3, m)$ must have a fault by Theorem 118. If the fault is vertical, we have $R(2, m)$ subtiling, which must have at least two flippable pairs (as we proved above). If the fault is horizontal, we have two rectangles with height at least two, and so neither has a peak and so each must have a flippable pair (Theorem 65), giving the total figure at least two flippable pairs. □

Theorem 101 (Kranakis (1996), Corollaries 7 and 8)**.** *For even squares $R(n, n)$, there are*

(1) *A unique tiling (up to rotation) that has 1 flippable pair.*

(2) *A unique tiling (up to rotation) that has 2 flippable pairs.*

[Not referenced]

Proof. We showed that any rectangle has at most 2 tilings with 1 flippable pair, and they are rotations of each other (Theorem 100). (This first part is a different proof from the one given in Kranakis (1996)).

For the second part, we consider several cases.

(1) Suppose there are 4 vertical dominoes on the border. Some of these may be adjacent to form groups.

  (a) If there are 4 groups, there are 3 flippable pairs by Theorem 96.

  (b) If there are 3 groups, one group has a flippable pair, and there are 2 additional flippable pairs by 96.

  (c) If there are 2 groups:

    i. If one group is not in the corner, we have one flippable pair in one of the groups, one flippable pair between them (Theorem 96), and one flippable pair between the one group and rectangle border (97).

ii. If both groups are in the corner, and both are flippable pairs, we have a third flippable pair between them (Theorem 96). If they are not both flippable pairs, then one is a flippable pair, and two more flippables are forced on the diagonals shown in Figure 92.

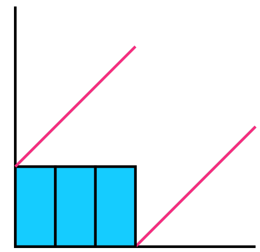


Figure 92: Two additional flippable pairs are forced.

In all cases, if we have 4 vertical dominoes, there are at least 3 flippable pairs, so this situation cannot occur.

(2) Suppose then there are 2 vertical dominoes.

(a) If there is one group, we have a flippabe pair. If it is not in the corner, we have two additional flippable pairs by Theorem 97, if it is, we have two additional flippable pairs on the diagonals as shown in Figure 93.

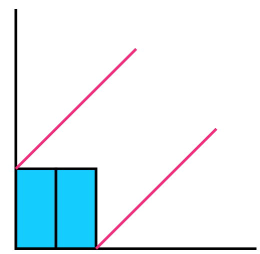


Figure 93: Two additional flippable pairs are forced.

(b) If there are two groups

i. If neither is in the corner, we have a flippable pair between them, and between each and a rectangle border, giving three in total.

ii. If one domino is in the corner, we have have one flippable pair between them, and one between the other and the rectangle border. There is a third flippable pair forced by the top corners of the rectangle, and this flippable pair cannot coincide with either of the other two. See Figure 94.

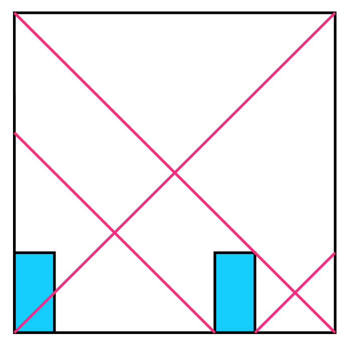


Figure 94: 3 forced flippable pairs.

This only leaves the case where there is a vertical domino in each corner.

The final case if there are no vertical dominoes.

We have shown that if there are exactly two flippable pairs, we either need no vertical dominoes, or there are exactly two, one in each corner. This can only be realized in one of two ways.

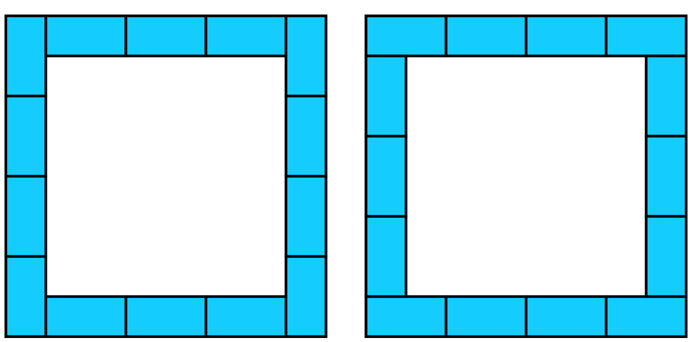
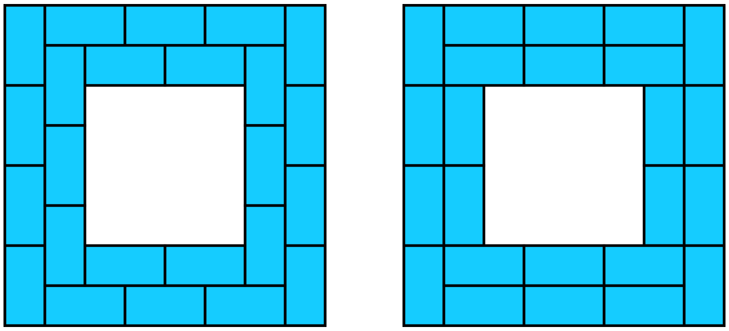


Figure 95: Two ways to complete the border.

In either case, the two flippable pairs are in a sub-square $R(n-2, n-2)$, and we can repeat the process above on this sub-square.

Going forward, we only have one option to build the border each time.



(a) A legal way to complete the border in the next step.

(b) Not a legal way to complete the border in the next step.

Figure 96: The next step.

Eventually, we arrive at one of the two tilings of the 4×4 square shown in Figure 97. This shows there are only two tilings with 2 flippable pairs, and they are 90 degree rotations of each other.

□

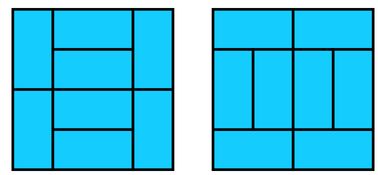


Figure 97: Two ways in which a 4×4 square can have exactly two flippable pairs.

In Kranakis (1996, Theorem 9) the following claim is made: *In a tiling of a $n \times n$ square with exactly 3 non-overlapping squares, there is a 8×8 square in the center that contains the three non-overlapping squares.* Although a proof is given the claim is in fact false, as the counterexample in Figure 98 shows. This pattern can be also constructed for larger squares, showing there is no bound on the subregion that contains the three squares.

The author also conjectures that in tilings with exactly four squares, either all the tilings lie within a bounded square, or the tiling can be partitioned into four squares with has a flippable pair in the center of each. This is also false, as shown in Figure 99.

It *does* seem plausible that in a tiling with more than one flippable pair, there is not a square larger than $R(n/2, n/2)$ that contains only one flippable pair.

Theorem 102 (Kranakis (1996), Theorem 10)**.** *A domino tiling of a rectangle $R(m,n)$, with $1 < m \leq n$ must have at least $\lfloor \frac{n}{m+1} \rfloor$ non-overlapping flippable pairs.*

[Not referenced]

Proof. The argument is similar to that used for Theorem 99, using Theorems 96 and 97. □

Figure 100 shows several tilings that achieves the minimum.

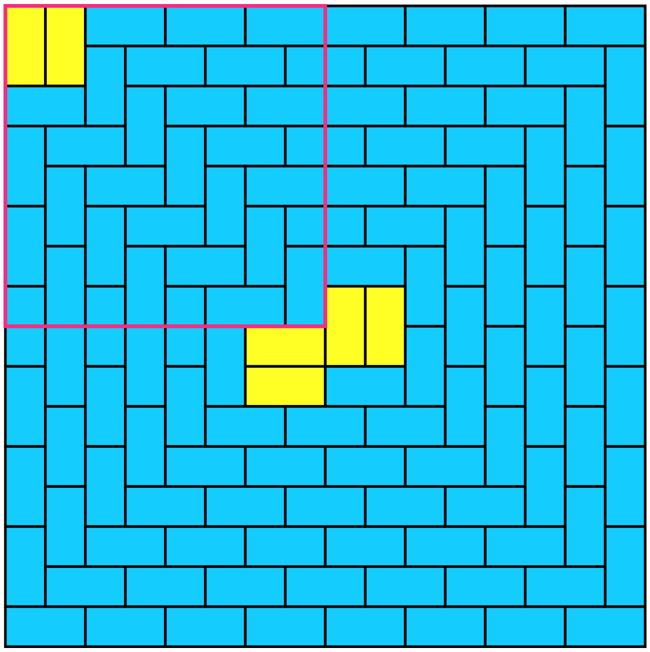


Figure 98: A counterexample to Kranakis's Theorem.

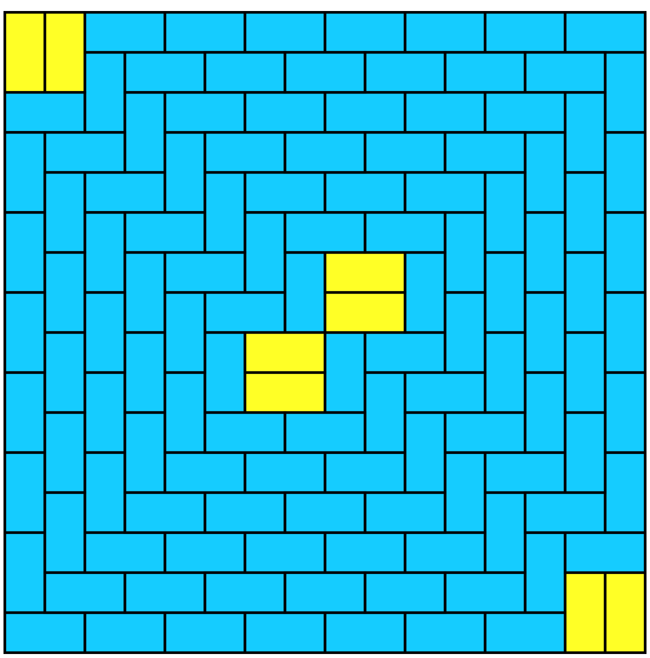


Figure 99: A counter example to Kranakis's Conjecture.

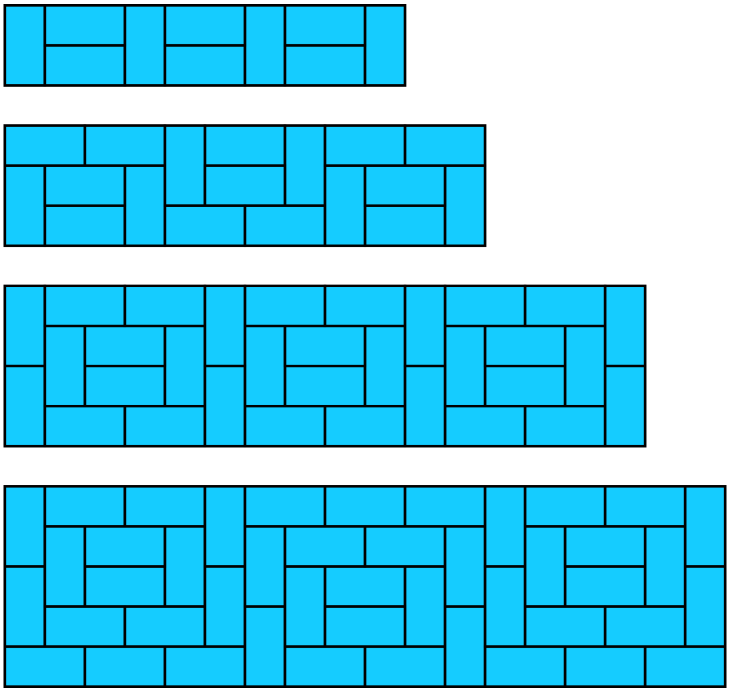


Figure 100: Tilings that achieves the minimum number of flips.

4.2 *Counting Tilings*

We have already seen a few results about the number of tilings of certain regions:

- Snakes have a unique tiling.
- Rings have two tilings.
- Any region with a 2×2 square as subregion has at least two tilings.

In this section we look at the number of tilings of rectangles and Aztec diamonds. We will denote the number of tilings of a region R by a tileset $\mathcal{T}$ as $\#_{\mathcal{T}} R$, or if $\mathcal{T}$ is understood, simply $\#R$.

4.2.1 *Rectangles*

We will use the notation $R(m, n)$ for an $m \times n$ rectangular region.

Theorem 103. *Let F_n be the Fibbonacci numbers* $1, 1, 2, 3, 5, 8, \cdots$ *(**A000045**).*

$$\#R(2, n) = F_{n+1} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right). \quad (4.6)$$

[Referenced on page 100]

Proof. There are two ways in which to start a tiling: with a vertical domino, or two horizontal dominoes.

The first case leaves us with a $R(2, n-1)$ and the second case leaves us with a $R(2, n-2)$. From this we find the recursion: $\#R(2, n) = \#R(2, n-1) + \#R(2, n-2)$. And by checking that $\#R(2, 1) = 1$ and $\#R(2, 2) = 2$, we are lead to the familiar Fibonacci sequence. That is, $\#R(2, n) = F_{n+1}$. See Knuth et al. (1989, Section 6.6) for how to derive the expression for Fibonacci numbers. □

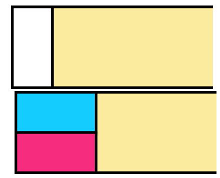
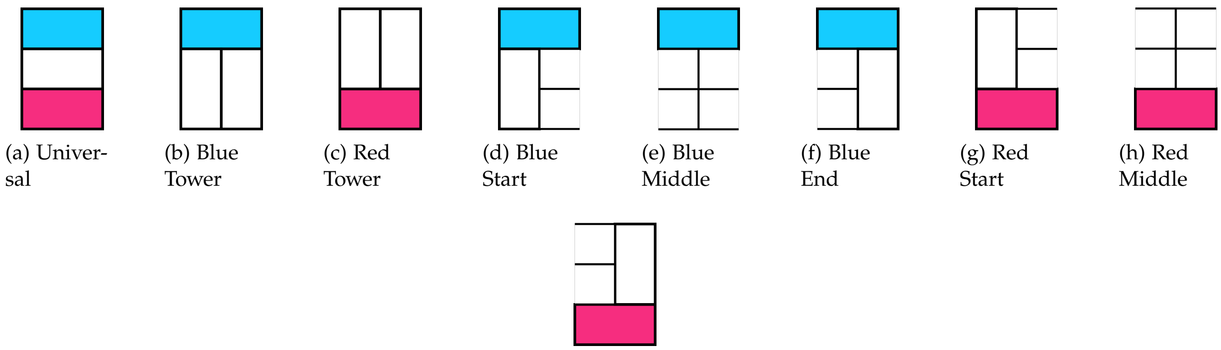


Figure 101: The two ways a domino tiling of $R(2, n)$ can start.

With a bit more work, we can also find a recursion for $R(3, n)$.

First, n must be even. That means we can divide the rectangle into $R(2, 3)$ blocks, and the only possibilities of how they are covered with dominoes are the ones shown in Figure 102.



(a) Universal (b) Blue Tower (c) Red Tower (d) Blue Start (e) Blue Middle (f) Blue End (g) Red Start (h) Red Middle (i) Red End

Figure 102: The only blocks that can make up $R(3, n)$. The terminology is from Butler et al. (2010)

Theorem 104.

$$\#R(3, n) = \frac{3+\sqrt{3}}{3}\left(2+\sqrt{3}\right)^{n/2} + \frac{3-\sqrt{3}}{3}\left(2-\sqrt{3}\right)^{n/2}$$

[Referenced on page 100]

Proof. Let S_n be the number of tilings possible for an $3 \times n$ rectangle starting with either the blue or red start piece. We can only start a rectangle with the universal piece, one of the towers, or one of the start pieces. This gives us:

$$\#R(3, n) = 3\#R(n-2) + 2S_n.$$

For a rectangle starting with a start piece, the only next piece is either the corresponding middle piece, or the corresponding end piece. This gives us:

$$S_n = \#R(3, n-4) + \#R(3, n-6) + \ldots + \#R(3, 2) + 1$$

Popping this into the equation above gives us:

$$\begin{aligned}
\#R(3,n) =& 3\#R(3,n-2) + 2(\#R(3,n-4) + \#R(3,n-6) + \ldots + \#R(3,2) + 1) \\
\#R(3,n) - \#R(3,n-2) =& 3\#R(3,n-2) + 2(\#R(3,n-4) + \#R(3,n-6) + \ldots + \#R(3,2) + 1) \\
& - 3\#R(3,n-4) - 2(\#R(3,n-6) + \#R(3,n-8) + \ldots + \#R(3,2) + 1) \\
=& 3\#R(3,n-2) - \#R(3,n-4)
\end{aligned}$$

Which gives us:

$$\#R(3,n) = 4\#R(3,n-4) - \#R(3,n-4) \tag{4.7}$$

Solving[7] this recurrence gives us:

[7] See Further Reading for references on how to do this.

$$\#R(3,n) = \frac{3+\sqrt{3}}{6}\left(2+\sqrt{3}\right)^{n/2} + \frac{3-\sqrt{3}}{6}\left(2-\sqrt{3}\right)^{n/2}$$

□

We get the sequence for even n: 3, 11, 41, 153, 571, ... This is **A001835** (skipping the first two terms).

The number of tilings of $R(3,n)$ *without* the universal tile is given by $2 \cdot 3^{n/2-1}$ for even n (Butler et al., 2010).

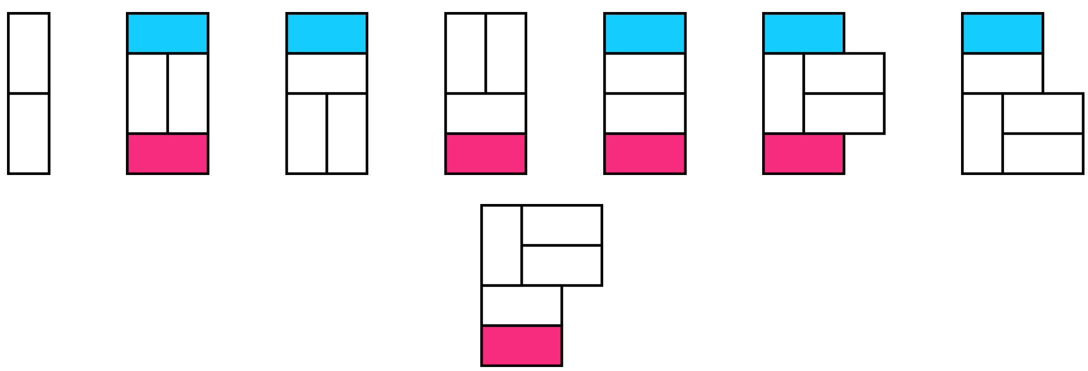


Figure 103: All the ways $R(4,n)$ can begin.

There are eight ways in which an $R(4,n)$ tiling can begin, shown and named in the picture below. We'll count the number with each kind of beginning configuration and add the results up to get $\#R(4,n)$. This gives us

$$\#R(4,n) = \#R(4,n-1) + 5\#R(4,n-2) + \#R(4,n-3) - \#R(4,n-4), \tag{4.8}$$

with $\#R(4,0) = 1$, $\#R(4,1) = 1$, $\#R(4,2) = 5$, and $\#R(4,3) = 11$.

How this occurrence is derived is shown by Kass (2014). An explicit formula is given in the entry for this sequence: **A005178**.

From 5 onwards it becomes very tedious to do this type of recurrence derivation by hand. Figure 104 lists the 25 ways a $5 \times n$ rectangle can begin.

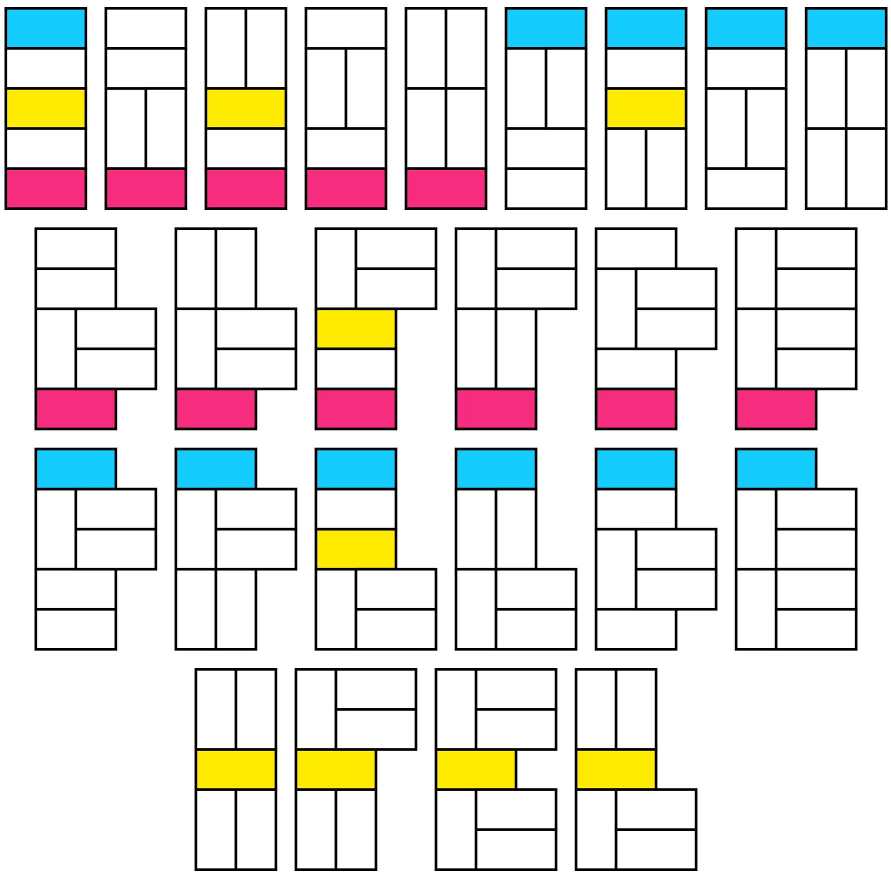


Figure 104: All the ways a $n \times 5$-rectangle can begin.

Theorem 105 (Kasteleyn (1961)). *The number of tilings of a $m \times n$ rectangle is given by*

$$\#R(m,n) = \prod_{k=1}^{m} \prod_{l=1}^{n} 2\sqrt{\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{l\pi}{n+1}} \qquad (4.9)$$

[Not referenced]

The proof uses advanced ideas and is lengthy, so I omit it. For a readable exposition of this theorem, see Stucky (2015). Table 9 tabulates some values. Showing that this theorem is equivalent to Theorems 103 and 104 for $m = 2$ and $m = 3$ is a tricky exercise.

A099390	1	2	3	4	5	6	7	8	9	10	11
1	0	1	0	1	0	1	0	1	0	1	0
2	1	2	3	5	8	13	21	34	55	89	144
3	0	3	0	11	0	41	0	153	0	571	0
4	1	5	11	36	95	281	781	2245	6336	18061	51205
5	0	8	0	95	0	1183	0	14824	0	185921	0
6	1	13	41	281	1183	6728	31529	167089	817991	4213133	21001799
7	0	21	0	781	0	31529	0	1292697 0	53175517	0	
8	1	34	153	2245	14824	167089	1292697	12988816	108435745	1031151241	
9	0	55	0	6336							
10	1	89	571								
11	0	144									

Table 9: Number of domino tilings of a $m \times n$ rectangle.

Problem 32. *How many tilings does $B(m^2 \cdot 2^n)$ have? (See Figure 105.)*

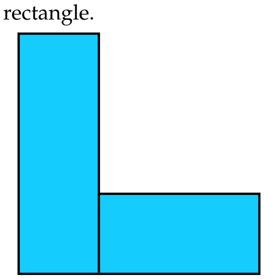


Figure 105: An example of a region with an L-shape composed from two rectangles of width 2.

4.2.2 *Aztec Diamond*

An **Aztec diamond** $A(n)$ is a region defined as follows:

(1) $A(1) = R(2,2)$

(2) To form $A(n+1)$, attach a neighboring cell to each open edge of $A(n)$.

It is easy to see that the area of an Aztec diamond is given by $2n(n+1)$ (if you divide the Aztec diamond into quarters, each quarter is a triangle with $1+2+3+n\cdots = n(n+1)/2$ cells).

Each row of an Aztec diamond is a strip polyomino with an even number of (connected) cells, and is therefor tileable by dominoes (Theorem 51). In fact, Aztec diamonds are strip polyominoes themselves.

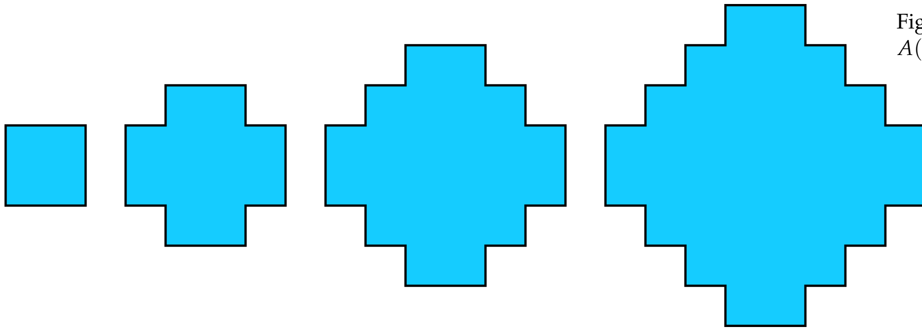


Figure 106: Aztec diamonds $A(1)$, $A(2)$, $A(3)$ and $A(4)$.

Problem 33. *Prove that Aztec diamonds are*

(1) *strip polyominoes*

(2) *Saturnian*

The number of tilings of $A(n)$ has been determined, but a proof of this is difficult, and I won't give it here. Instead, I will prove some other interesting properties.

Theorem 106. *Let R be an Aztec diamond, and S a subregion of R also an Aztec diamond with radius one less, sharing some border with R. Then in any tiling of R, either 0 or 2 dominoes cross the border of S.*

[Not referenced]

Proof. Since S and $R - S$ is tileable, it is possible that a tiling of R has no dominoes that cross the border of S. Suppose one cross the border, and that the cell in S covered by that domino is white. Another domino must cross with the cell in S covered by it black (to keep the balance). So if one domino cross, then at least two must cross.

Any dominoes that crosses the border with the same color cells in S, must lie on the same side of S, and therefor two with no other border-crossing dominoes between them demarcates a snake with endpoints of the same color, which then has odd area (Theorem 50) and is untileable (Theorem 1). □

Figure 108 shows an example. Note also that most dominoes in the subregion is aligned with similar orientation. This partially explains the Arctic behavior we discuss in the next subsection.

Let $A(n,i,j)$ be an Aztec diamond with two cells on the border removed in columns i and j as in the theorem above. We can then write:

$$\#A(n+1) = \#A(n) + 2\sum_{i=1}^{n}\sum_{j=n+1}^{2n} \#A(n,i,j). \qquad (4.10)$$

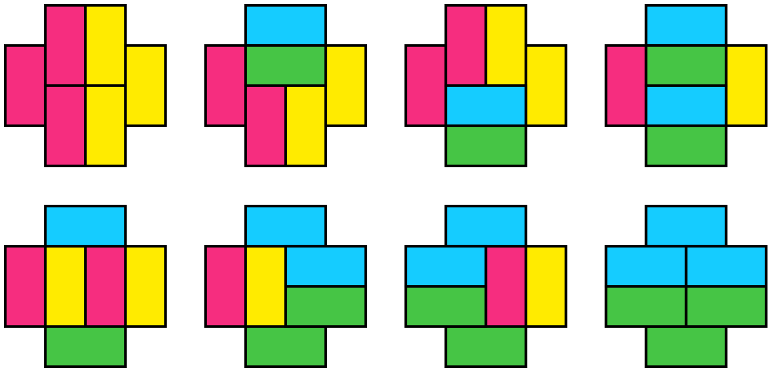


Figure 107: The 8 tilings of AD(2)

Theorem 107. *The number of domino tilings of an Aztec diamond $A(n)$ is given by*

$$\#A(n) = 2^{\binom{2}{n}} = 2^{\frac{n(n+1)}{2}}$$

[Not referenced]

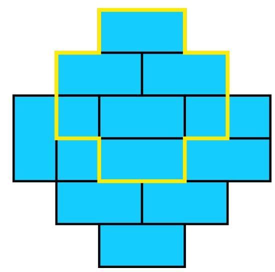


Figure 108: At most two dominoes can cross the border of the subregion.

The theorem was first proved in Elkies et al. (1992a) and Elkies et al. (1992b) (they give four different proofs). Simpler proofs are given in Eu and Fu (2005) and Fendler and Grieser (2016), with more details provided in Simon (2016). Table 10 tabulates some values.

Table 10: Number of tilings of the Aztec diamond.

n	$\|A(n)\|$ A046092	$\#A(n)$ A006125
0	0	1
1	4	2
2	12	8
3	24	64
4	40	1024
5	60	32768
6	84	2097152
7	112	268435456
8	144	68719476736
9	180	35184372088832
10	220	36028797018963968
11	264	73786976294838206464
12	312	302231454903657293676544
13	364	2475880078570760549798248448
14	420	40564819207303340847894502572032

4.2.3 Arctic Behavior

In Jockusch et al. (1998) the authors proved that randomly generated tilings of large Aztec diamonds had some peculiar patterns in the corners: the dominoes tend to be oriented in specific ways. This phenomenon does not occur for rectangles, where the orientation of dominoes are arbitrary.

Figures 109 and 110 show examples.[8]

Why is this happening? The formal statement and proof of this theorem (called the *Arctic Circle Theorem*) is beyond the scope of this essay. But we can get of an intuitive understanding of what is going on. We have already seen that in certain subregions next to the border, dominoes tend up to line a certain way (Sections 4.1.2 and 3.2.2). Another way to look at this phenomenon is to look at the flow. Divide the region into four partitions as shown. In each, we calculate the deficiency as $\frac{2(n+1)}{4}$. That means, we need as many dominoes to cross over the axes. The axis has $2\left\lfloor\frac{n}{2}\right\rfloor+1$ crossing points. A quick calculation shows in this setup

$$c(n) = \begin{cases} 2d(n)+1 & \text{if } n \text{ is even} \\ 2d(n)-1 & \text{if } n \text{ is odd.} \end{cases}$$

This means about at least half the potential crossing points need to be actually crossed. Now when two such dominoes lie next to each other, the also determine the orientation of two dominoes between them. If there are three next to each other, then they determine the orientation of 6; in general the more crossing dominoes lie next to each other, the more other dominoes' orientation are forced. Because there is a large amount of dominoes that need to cross relative to the number of crossing points, on average there will be plenty of groups of adjacent dominoes oriented the same way, and therefor, on average, dominoes in this region tend to line up.

With this type of reasoning, we can see that this type of behavior will be typical of regions with reasonably sized stretches of border of the same color.

[8] The software used to generate this software is available online; see Doeraene.

4.3 Constraints and Monominoes

4.3.1 Optimal Tilings

If a region is not tileable, how close can we get to tiling it? Asked differently: Given a set of dominoes and monominoes, what is the least number of monominoes required to tile the region?

Let $\mathcal{T}$ be a tileset without any monominoes, and let $\mathcal{T}^+$ be the tile set with a monomino added. Given some region R to tile with $\mathcal{T}^+$,

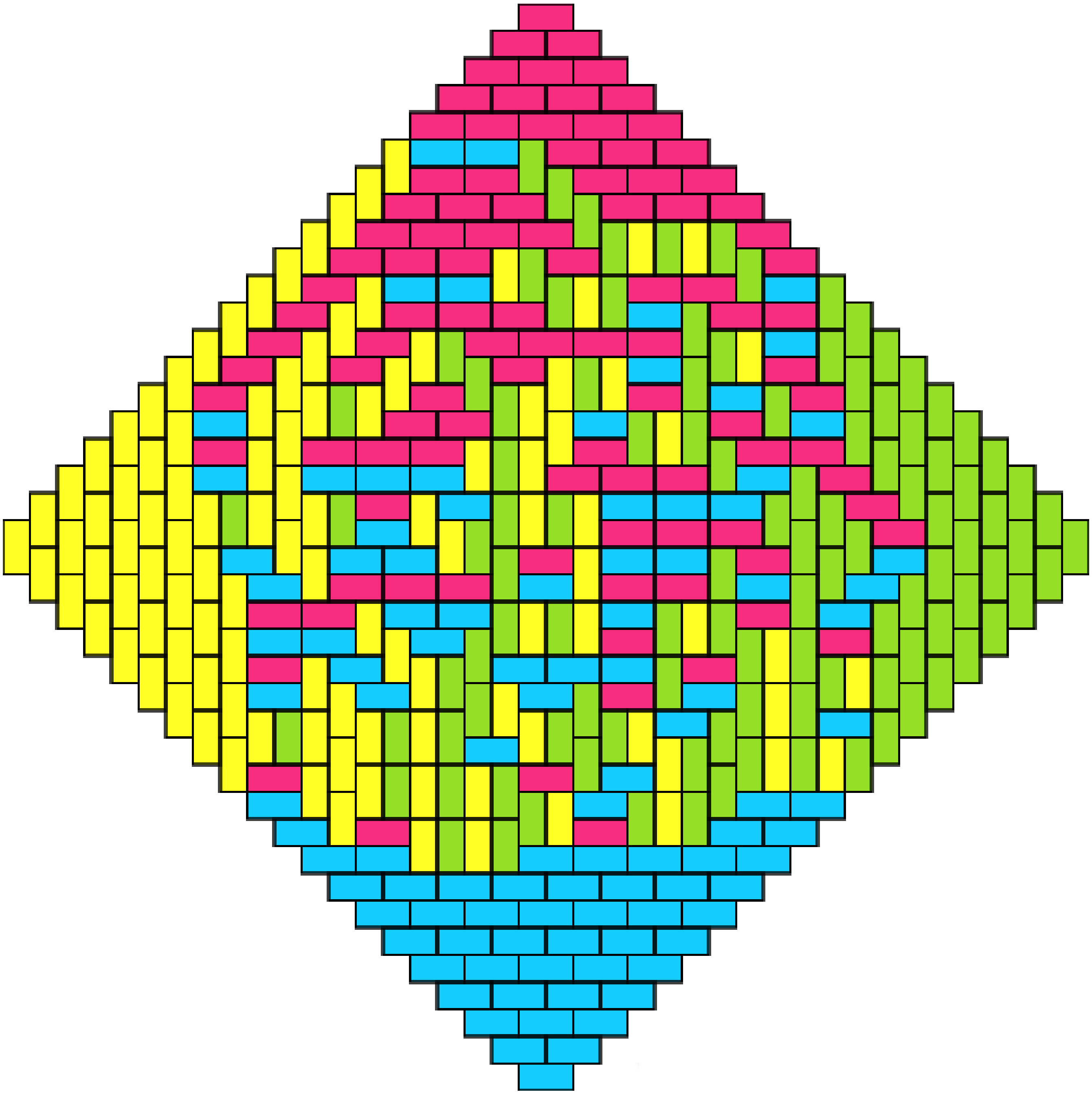


Figure 109: A tiling of an Aztec diamond.

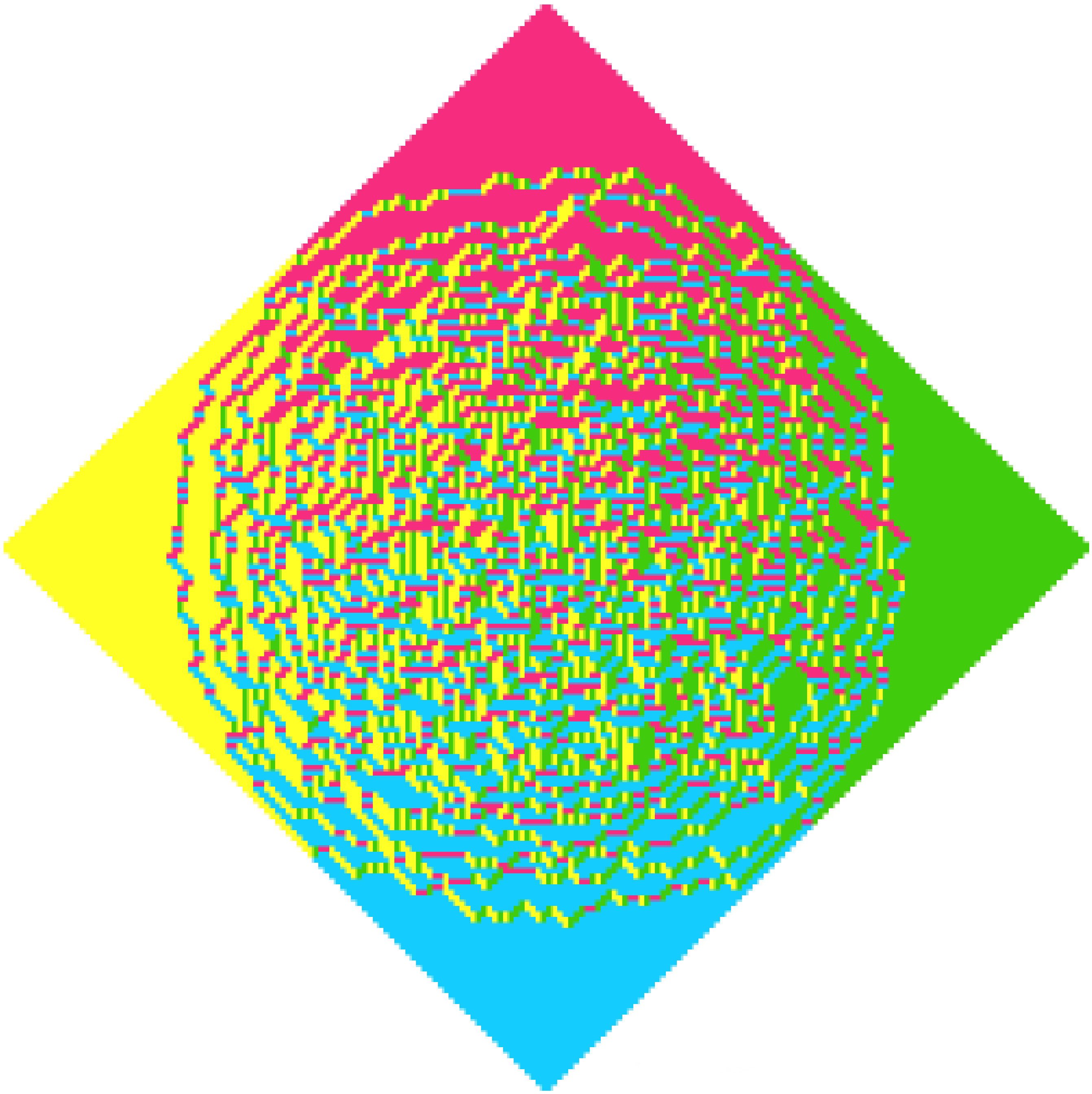


Figure 110: A tiling of an Aztec diamond.

the least number of monominoes we require is called the **gap number** of the region, with respect to the tileset $\mathcal{T}$ (Hochberg, 2015), and we will denote it by $G_{\mathcal{T}}(R)$, or simply $G(R)$ if the tile set is clear. (In this section, $\mathcal{T}$ is always the set containing the domino.) If $G(R) = 0$, then R is tileable.

A tiling of R that uses exactly $G(R)$ monominoes is called **optimal** (following Bodini and Lumbroso (2009)).

Theorem 108. *For dominoes, the gap number must at least be the absolute deficiency:*

$$G(R) \geq |\Delta(R)|.$$

[Referenced on pages 107 and 119]

Proof. Let M be the set of cells that are covered by monominoes in a optimal tiling of R. Then $R - M$ is tileable by dominoes, so $\Delta(R - M) = 0$ (Theorem 23), and $|\Delta(M)| \leq |M| = G(R)$ (See Problem 7.3). But $|\Delta(R)| \leq |\Delta(R - M)| + |\Delta(M)|$ (Problem 7.4), so $|\Delta(R)| \leq G(M)$. □

Theorem 109. *For a tileset $\mathcal{T}$ where $|P| = n$ for all $P \in \mathcal{T}$, we have*

$$G_{\mathcal{T}}(R) \equiv |R| \pmod{n}. \tag{4.11}$$

[Referenced on page 106]

Proof. If $G_{\mathcal{T}}(R) = g$, then there is a tiling with tiles from $\mathcal{T}$ and g monominoes. Remove the cells covered by monominoes to form R'. Now since R' is tileable by $\mathcal{T}$, we know that $|R'| \equiv 0 \pmod{n}$ (Theorem 1). Thus $g + |R'| \equiv g \pmod{n}$, or $|R| \equiv g \pmod{n}$. □

Note that this theorem is stated for any tileset, not just for dominoes. For dominoes, it follows that $G(R) \equiv |R| \pmod{2}$.

Theorem 110. *If R has a tiling with dominoes and a single domino, then $G(R) = 1$, and so the tiling is optimal.*

[Referenced on page 106]

Proof. $G(R) \leq 1$ (since we have a tiling realized with one monomino), and it cannot be 0 since $|R|$ must be odd (Theorem 109), and so $G(R) = 1$. □

Theorem 111. *Strip polyominoes with odd area has gap number 1.*

[Not referenced]

Proof. Put the monomino on the first cell of the strip polyomino. The remainder of the figure is a strip polyomino with an even number of cells, and therefor tileable by dominoes (Theorem 51). We have a tiling with single monomino, and so the gap number must be 1 (Theorem 110). □

Theorem 112. *Monominoes in optimal domino tilings cannot be neighbors.*

[Referenced on page 110]

Proof. Any two neighboring monominoes can be replaced with a domino, and so the original tiling cannot be optimal. □

Theorem 113. *All optimal tilings of a region have the same number of black, and the same number of white monominoes.*

[Not referenced]

Proof. Suppose we have two tilings, T with W white monominoes and B black monominoes, and T' with W' and B' white and black dominoes respectively, and let S be the subregion covered by dominoes in T, and S' be the subregion covered by dominoes in T'.

Now we have $\Delta(R) = \Delta(S) + B - W = \Delta(S') + B' - W'$. But S and S' are tileable by dominoes, so $\Delta(S) = \Delta(S') = 0$ (Theorem 23). So $B - W = B' - W'$. But also, since the tilings are minimal, we have $B + W = B' + W'$, which gives is $B = B'$ and $W = W'$. □

Class	$G(R)$
Strip polyomino with odd area (includes snake, rectangle)	1
Column convex polyomino with k odd columns (includes cylinder)	$\leq k$
Bar graph	See Bodini and Lumbroso (2009)
Young diagram (includes L-shaped polyomino, rectangle)	$\lvert\Delta(R)\rvert$

Table 11: Gap numbers for some classes of polyominoes.

Theorem 114. *The gap number of a Young diagram R is $|\Delta(R)|$.*

[Not referenced]

Proof. Suppose we cannot remove a cylinder from the region. Then, the Young diagram must have top cells in every column the same color (Theorem 40). If we put a monomino on each of the k top cells of odd columns, the remainder can be tiled with dominos. Clearly $|\Delta(R)| = k$, so the gap number must be at least k (Theorem 108) and since we can find a tiling with k monominoes and dominoes, the gap number is exactly k.

Suppose we *can* remove a cylinder. We can continue this until we cannot. We now have the shape described above. Find a tiling for it, and note that the absolute value of the deficiency is equal to the number of monominoes. It remains to show the original figure can be tiled with the required monominoes.

Now re-insert cylinders that were remove (using the process of Theorem 37) to find a tiling of the original region k monominoes. This means $G(R) \leq k$. But $G(R) \geq |\Delta(R)| = k$, therefor $G(R) = k = |\Delta(R)|$. □

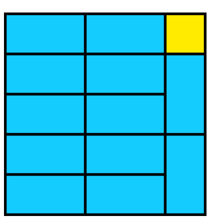
Once we have determined the gap number, we can also ask: where are legal positions for these monominoes to occur?

Example 13. *In a rectangle with odd area, the deficiency is 1, and say WLG that $B > W$. Then at the very least the monomino must be placed on a black square. It is easy to see that a monomino can occur on the corner of such a rectangle. Using this, we can show it is possible for this monomino to occur on* any *black cell.*

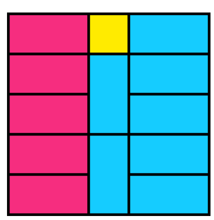
First, suppose we want to place a monomino somewhere on border of the rectangle. We can now divide the rectangle into two rectangles such that one is even, and the other is odd with the monomino in the corner. Note that we can only do this when the monomino is placed on a black square, as is required. Both these are tileable, and so is the whole.

Suppose then the monomino is somewhere in the interior of the rectangle. Again, we can divide the rectangle into two rectangles, one which is even, and one which is odd with the monomino on the border. And again we can only do this because the monomino has to be placed on a black square. Both these are tileable, and therefor so is the whole.

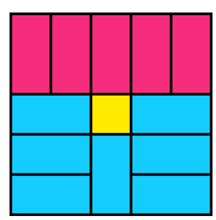
Table 12 tabulates the number of tilings for some rectangles with odd area by dominoes and a single monomino.



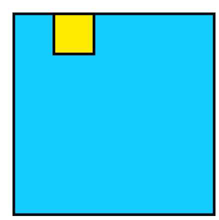
(a) A rectangle with odd area and a monomino placed in the corner is tileable.



(b) A rectangle with odd area and a monomino placed on a black cell the edge is tileable.



(c) A rectangle with odd area and a monomino placed on a black cell in the interior is tileable.



(d) A rectangle with a monomino on a white cell is not tileable, since the remainder of the region is unbalanced.

It is not always the case that monominoes can occur on any cell of the correct color. In Figure 111 for example, there is no optimal tiling of the region with the monomino on the red cell. In this case removing the red cell partitions the remaining region into two regions with an odd area each, so neither is tileable by dominoes (Theorem 1). The same applies if the cell would partition the region into two regions and one is unbalanced, as shown in Figure 112.

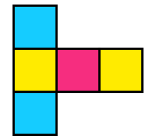


Figure 111: There is no optimal tiling of this region with the monomino covering the red cell.

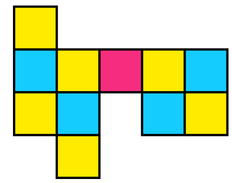


Figure 112: The gap number of this region is 1, but there is no optimal tiling with the monomino covering the red cell.

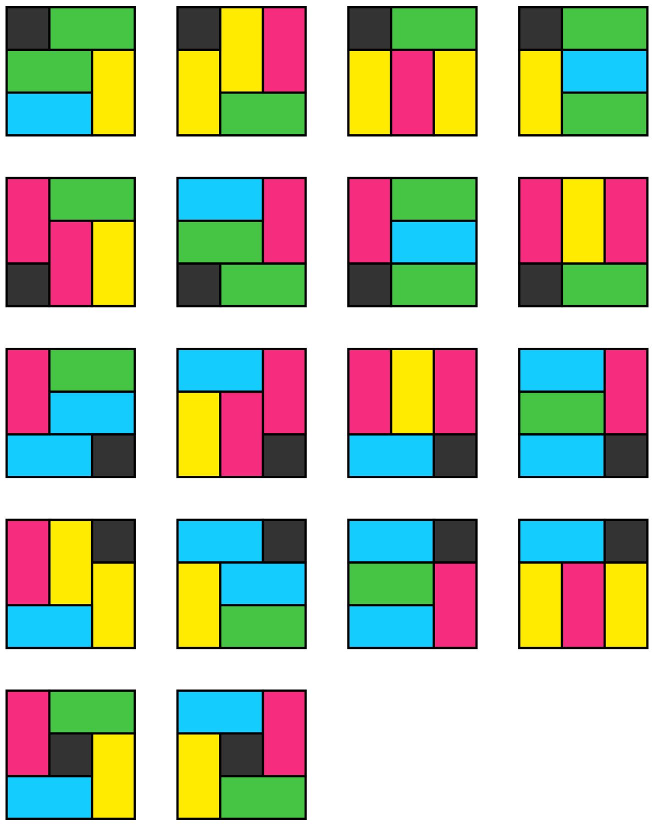


Figure 113: The optimal tilings of a 3×3 square.

	1	3	5	7	9
1	1	2	3	4	5
3	2	18	106	540	2554
5	3	106	2180	37924	608143
7	4	540	37924	2200776	116821828
9	5	2554	608143	116821828	

Table 12: Number of tilings of odd-area rectangles with one monomino and dominoes.

The following Theorem generalizes Theorem 61.

Theorem 115. *Suppose R has two tilings T and U by dominoes and monominos. Then for each monomino in T that is not in U, there is a strip S that is tiled by a set of dominoes and a monomino in each tiling, and the monomino is at one of the strips ends in each tiling.*

[Referenced on page 110]

Proof. Let C_1 be a cell tiled by the monomino in T. That cell is covered by a domino in U, and has neighbor in that domino C_2. C_2 is covered by a domino in T, it's neighbor is C_3. We can continue this process until we reach C_k, which covers a monomino in U.

Note the following:

- k must be odd (this can be seen from the process itself, or from the fact that the two monominoes must have the same color, and hence any strip of which they are the endpoints must have an odd number of cells).

- $C_i \neq C_j$ if $i \neq j$. For suppose $C_i = C_j$ and $i < j$. It follows that $C_{i-1} = C_{j-1}$, and so on until we have $C_1 = C_{j-i+1}$. But unless $i = j$, it means that C_{j-i+1} is not covered by a domino in T (it is covered by a monomino), and therefor cannot have been part of the sequence in the first place, because all cells in the strip $C_{i\neq 1}$ are covered by dominoes in T.

□

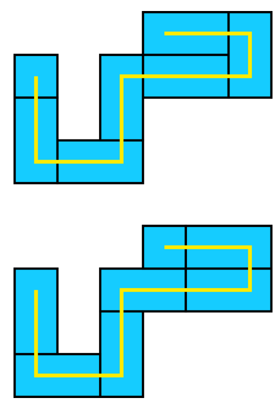


Figure 114: An example of a strip shift.

Suppose that T is a tiling of R, and that $C_1, \cdots C_k \in R$ is a strip, and that C_1 is covered by a domino in T, and C_{2i}, C_{2i+1} is covered by a single domino in T. We can now form a new tiling T' by covering each C_{2i-1}, C_{2i} by a domino, and C_k by a monomino. This transformation is called a **strip shift**. See Figure 114 for an example.

The following Theorem generalizes Theorem 63.

Theorem 116. *Any tiling of a region R by dominoes and k monominoes can be transformed into any other tiling with dominoes and k monominoes by performing strip rotations and strip shifts.*

[Referenced on page 110]

Proof. Let T and U be two tilings of a R. For each monomino in T not in U, there is a strip S_i such that if we transform a strip shift in T, we get the strip's tiling in U (Theorem 115). Let M_i be the cells covered by monominoes in both T and U. Then $R - \bigcup_i S_i - \bigcup_i M_i$ is a region with subtilings in T and U, that is covered by only dominoes in both tilings. These tilings are connected by strip rotations (Theorem 63).

□

Here are some consequences of Theorems 115 and 116:

(1) In an optimal tiling, no two monominoes can lie in the same tiled strip. This is a generalization of Theorem 112.

(2) A region R cannot be partitioned into strips so that fewer than $G(R)$ of them have odd length.

4.3.2 *Tatami Tilings*

[9] A **tatami tiling** is a tiling such that no four corners of tiles meet (Knuth (2009, Problem 7.1.4.215); *see also* Ruskey and Woodcock (2009) and Hickerson (2002)). Notice that this definition does not reference any tileset. There are two types of tatami tilings that are usually considered: tatami tilings with dominoes, and tatami tilings with dominoes and monominoes.

[9] This is a placeholder section which I hope to expand in a future version.

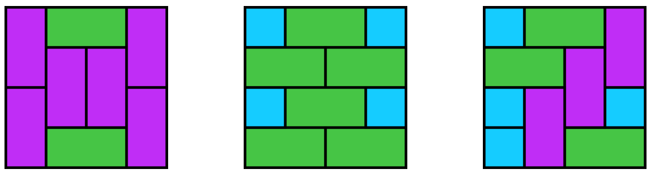
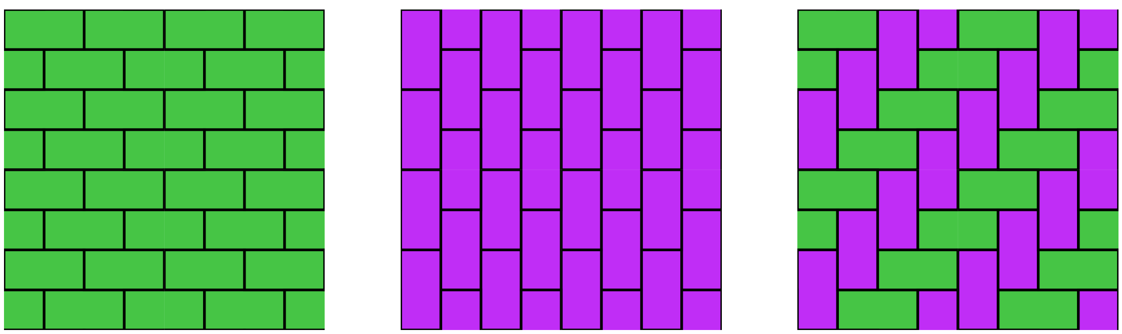


Figure 115: Tatami tilings of the 4×4 square with dominoes and monominoes.

The plane has many tatami tilings with dominoes; several are shown in Figure 116.



(a) Horizontal Bond Pattern (b) Vertical Bond Pattern (c) Herringbone Pattern

Figure 116: Tatami tilings of the plane.

4.3.3 *Fault-free tilings*

A **fault** in a tiling is a line on the grid (horizontal and vertical) that goes through the region and is not crossed by any tile (Golomb, 1996, p. 18).[10] See Figure 117 for an example. A tiling with no faults are called **fault-free**. We are mostly interested in fault-free tilings of *rectangles*. See Figures 118 and 119 for examples.

[10] Also called *line of cleavage*, Reid (e.g. 2005). The problem of finding fault-free tilings was first proposed by Robert I. Jewett according to Golomb.

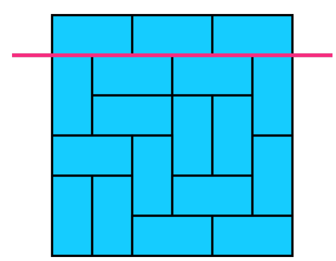


Figure 117: A tiling with a horizontal fault.

Theorem 117 (Golomb (1996), p.18)**.** *Domino tilings of* $2 \times n$ *rectangles cannot be fault-free.*

[Referenced on page 114]

Proof. There are only two configurations in which a rectangle can start (Figure 101). Both of these have faults. □

Theorem 118 (Golomb (1996), p.18 (no proof))**.** *Domino tilings of* $3 \times n$ *rectangles cannot be fault-free.*

[Referenced on pages 92 and 114]

Proof. A rectangle can only start with a universal block, a tower, or one of the start blocks (see Figure 102). The universal block and towers all have faults. A start block can only be followed by an end block, or a mid-block; and a mid-block can only be followed by another mid-block or an end block. So, since the rectangle is finite, if it starts with a start block, it must eventually have an end block. If that is the final block, the rectangle has a horizontal fault. Otherwise, it has a vertical fault. □

Theorem 119 (Golomb (1996), p.18)**.** *Domino tilings of* $4 \times n$ *rectangles cannot be fault-free.*

[Referenced on page 114]

Proof. (Martin, 1991, p. 18) All the ways a $4 \times n$ rectangle can begin is shown in Figure 103. The first 5 can also end the rectangle, or they can be extended. In the former case, each of those 5 has a fault. In the later case, there is a vertical fault at the end of the block.

The third last can be extended only by adding horizontal dominoes to the top and last row to the right. This shape can either be completed with a vertical domino, in which case there are two horizontal faults, or with two horizontal dominoes, which gives us the same situation as we started with (so eventually, a fault must be introduced).

The final two shapes can be extended by either a vertical domino, or two horizontal dominoes. In the former case, we have a horizontal fault, or if the rectangle continues, a vertical fault. The the latter case, we have the same situatiom we started with, so eventually a fault must be introduced. □

Theorem 120 (Golomb (1996), p.18)**.** *The* 5×6 *rectangle has a fault-free tiling by dominoes.*

[Referenced on page 114]

□

Theorem 121 (Golomb (1996), p.18)**.** *The* 6×6 *has no fault-free tiling by dominoes.*

[Referenced on page 114]

Proof.

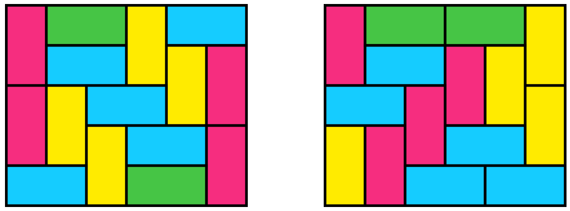


Figure 118: The two fault-free tilings of a 6×5 rectangle.

Proof. Consider any potential fault line. This line must be covered by a domino. If no other domino covers it, then the line divides the figure into two figures of odd area each (the area is odd since the cells covered by the fault-covering domino are not included). Then there must be at least one other domino on the line (Theorem 21). Since there are 10 lines, we need 20 dominoes. But the area is only 36, which means it will be covered by 18 dominoes. It is therefore impossible to cover all faults and still tile the figure. Thus, the figure has no fault-free tiling. □

Theorem 122 (Golomb (1996), p.19)**.** *A 6×8 rectangle has a fault-free tiling.*

[Referenced on page 114]

Proof.

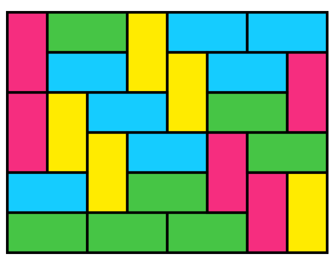


Figure 119: A fault-free tiling of a 6×8 rectangle.

□

Theorem 123 (Golomb (1996), p.19)**.** *If we have a fault-free tiling of a $R(m, n)$, we can find fault-free tilings of $R(m+2, n)$ and $R(m, n+2)$.*

[Referenced on pages 42 and 114]

Proof. The same argument works for both types of extensions, so let's prove it for horizontal extensions. Pick a vertical domino on the border. (Such a domino must exist, for if it does not, then there would be a fault.) Remove it from the tiling. Now add a vertical 2-cylinder of dominoes along the crooked edge. Add the removed domino in the opening. The new rectangle is 2 cells wider, and is still fault-free. To see this, note that all original fault lines are still covered. We introduced two new potential faults. The inner most is covered by the dominoes that fit into the opening of the crooked edge; the other is covered by the remaining dominoes. □

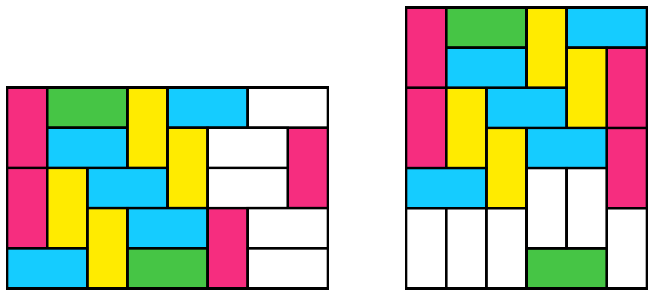


Figure 120: Extending a fault-free tiling of a 6×5 rectangle horizontally and vertically.

Theorem 124 (Golomb (1996), p.18)**.** *All even-area rectangles $R(m, n)$ (except $R(6, 6)$) has fault-free tilings by dominoes if $m, n \geq 5$.*

[Not referenced]

Proof. By Theorem 123, we can use suitable extensions to find fault-free rectangles for $R(5 + 2p, 6 + 2q)$ and $R(6 + 2p, 8 + 2q)$, since both 5×6 and 6×8 rectangles as has fault-free tilings (Theorems 120 and 122). Clearly a $1 \times m$ rectangle must have faults. Theorems 117, 118, and 119 shows we cannot have fault-free tilings when $1 < m \leq 4$ or $1 < n \leq 4$. And finally, Theorem 121 shows there are no fault-free tilings of the 6×6 rectangle. □

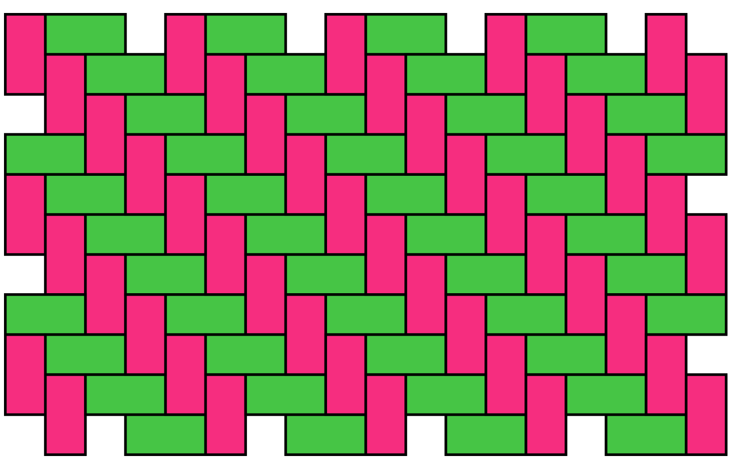


Figure 121: Fault-free tiling of the plane.

Problem 34.

(1) *What rectangles have fault-free tatami tilings with*

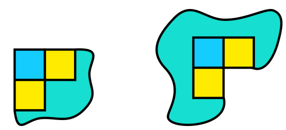
(a) *dominoes only,*

(b) *dominoes and monominoes?*

(2) *Prove any tiling of an Aztec diamonds has at least three faults.*

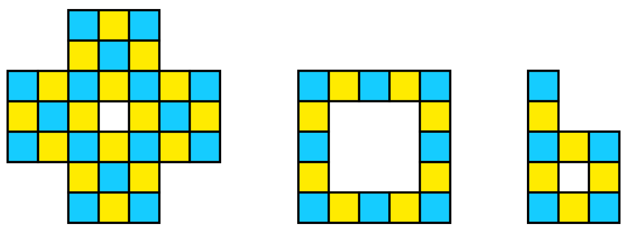
4.3.4 *Even Polyominoes*

An **even polyomino** is a polyomino where all corners (of inside and outside borders) neighbor black cells if we color the polyomino and its surroundings with the checkerboard coloring (Kenyon, 2000b). Note that the holes of an even polyomino are even polyominoes themselves.



(a) (b)

Figure 122: Valid corners in even polyominoes.



(a) (b) (c)

Figure 123: Examples of even polyominoes.

Theorem 125. *Let R be a even polyomino with H holes. Then*

$$\Delta(R) = 1 - H. \tag{4.12}$$

[Referenced on page 119]

Proof. For each edge E_i, notice that:

(1) if E_i is a mountain, it lies between two black cells, and hence $b(E_i) - w(E_i) = 1$,

(2) if E_i is a valley, it lies between two white cells, and hence $b(E_i) - w(E_i) = -1$, and

(3) if E_i is a flat, it lies between a black and white cell, and hence $b(E_i) - w(E_i) = 0$.

Therefor, we have for the entire polyomino, $b(R) - w(R) = P - V$. By Theorem 27 we have $\Delta(R) = \frac{b(R)-w(R)}{4}$, so $\Delta(R) = \frac{P-V}{4}$. And by Theorem 13 we have $\frac{P-V}{4} = 1 - H$, so $\Delta(R) = 1 - H$. □

It follows immediately that even polyominoes without holes have an odd area. Also note that in even polyominoes, mountains and valleys have odd length, and flats have even length.

Problem 35.

(1) *Show that if we insert a 2-cylinder into an even polyomino, the result is an even polyomino.*

(2) *If we remove a 2-cylinder from an even polyomino, is the result an even polyomino?*

Theorem 126. *Even polyominoes with one hole are tileable.*

[Referenced on pages 118 and 119]

Proof. For this proof to work, we will manipulate the border (either the outside or inside border) to give a new polyomino with a hole, and we will consider regions where borders overlap (but not cross) also valid polyominoes, and they are even if the corners are black as with the regular definition. Here are some examples of such polyominoes (the outside border has been slightly displaced where it coincides with the inside border).

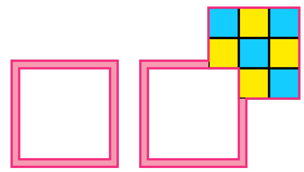


Figure 124: Border polyominoes.

(1) In an even polyomino, all peaks must be black (otherwise there are white corners). Also, all black peaks must be attached to a white cell with exactly two neighbors. Therefor, in an even polyomino, if we move the border to exclude the peak and its neighboring cell, we are still left with an even polyomino.

An example of this manipulation is shown in Figure 125.

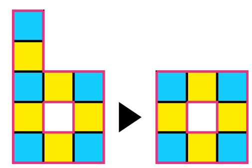


Figure 125: Peak removal

(2) A 2×2 square can be of two types. Type I has a black square in the top right, a Type II square has a black square in the top left. If a Type I square occurs in a corner of the polyomino, we can move the border to exclude it, and keep the polyomino even.

An example of this manipulation:

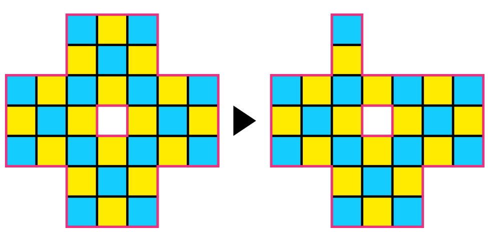
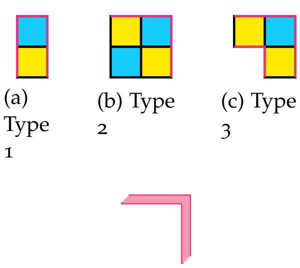


Figure 126: Corner removal

(3) All convex corners of a even polyomino are one of the four types shown in Figure 127.

(4) If we perform the manipulations in 1 and 2 until we cannot, we are only left with Type 1 and Type 2 corners.

(5) A region with only type 3 and 4 convex corners and one hole cannot contain a 2×2 subregion.



(a) Type 1 (b) Type 2 (c) Type 3 (d) Type 4

Figure 127: Corner Types

Suppose there is such a square. Find the largest connected set of cells that contains this square such that each cell is part of a 2×2 subregion. This set of cells must have at least four convex corners (Theorem 8). For these corners to *not* be corners of the entire region, each needs to be "covered" by a "strand" (the picture below shows a possibility.) These strands must either result in peaks, or they must connect in pairs. If they connect in pairs, there are two holes, and this is impossible (Figure 128). Therefor, there cannot be any 2×2 subregion.

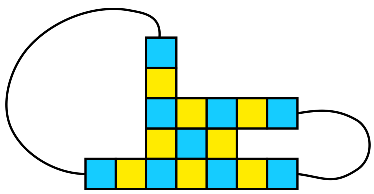


Figure 128: An example where strands that cover potential corners meet up in pairs, leading to two holes.

In this image, there is a 2×2 square, contained in a 3×3 square where each corner is covered by a white cell that stars the strand. In this example, the strands connect in pairs, leading to two holes.

(6) We cannot have any X- or T-junctions in a region with no 2×2 subregions, no peaks, and one hole.

Suppose we have an X-junction. Each of the four connectors must eventually lead to a peak, or pairs of them must join. In the latter case, we have two holes, which is impossible. Therefor there are no X junctions.

Suppose we have a T-junction. Each of the three connectors must either lead to a peak, or they must join other connectors. We can only join up if we have at least another T junction, in which case we will have at least have two holes, which is impossible. Therefor there are no T-junctions.

(7) This means we either have a ring, or a polyomino with no area (thus the inner and outer border coincide completely).

(8) This procedure shows we can partition any even polyomino with one hole into a set with dominoes, 2×2 squares, and either a ring (tileable by Theorem 51) or empty polyomino. All the elements of this partition are tileable, and therefor, so is the whole region (Theorem 2).

□

The double border definition is necessary to deal with polyominoes such shown in Figure 129. If we did not have that, removing a

2×2 corner would not leave an even polyomino. (It would be nice to find a proof that does not require this trick.)

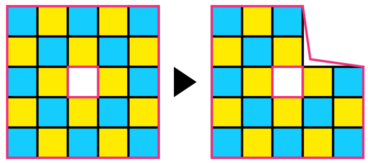


Figure 129: An example of a polyomino for which our proof would fail if we did not use the double border definition of a polyomino.

One consequence of using this formulation is that the hole need not be an actual hole using the normal definition of a polyomino.

An algorithm follows immediately from the border manipulations.

Theorem 127. *A rectangle with odd area with an odd hole is tileable if it is balanced.*

[Not referenced]

Proof. Let $R = R_1 - R_2$ be our region, and R_1 the filled rectangle, and R_2 the hole. The deficiency of an odd-area rectangle is 1 or -1. (You can pair rows to cancel out, leaving the last row with one more cell of one color than the other.) Suppose the corners of the rectangle is black. Then, for the region to be balanced, the corners of the hole must also be black, because we have $\Delta(R) = \Delta(R_1) - \Delta(R_2)$, or $0 = 1 - (B - W)$, So $B - W = 1$, and so we have $B > W$, which means the rectangle must have black corners. Therefor R is an even polyomino, and so is tileable by dominoes (Theorem 126). □

We can "fix" the deficiency of even polyominoes in a specific way to make a class of polyominoes that are tileable by dominoes.

A **Temperleyan polyomino** is a polyomino formed from an even polyomino by removing one black cell from the outer border, and appending a black cell to each inner border Kenyon (2000b).

Theorem 128. *Temperleyan polyominoes are tileable by dominoes.*

[Referenced on page 119]

Proof. *Case* $h = 0$. Suppose R is the polyomino, and H is the black cell removed from the border. Then let $R' = R \cup H$. We can now view this polyomino as a double border polyomino, with outer border the border of R', and the inner border the border of H. Since R' is even and H is an even hole, R must be tileable by Theorem 126.

General case. See Kenyon (2000b, Section 7) for a sketch of a proof.

□

Example 14. *Suppose R is a region partitioned into two simply-connected subregions S_1 and S_2, with S_1 a black even polyomino and S_2 a white even polyomino. Then R is tileable. Let $v_1 \in S_1$ and $v_2 \in S_2$ be neighbors with v_1 black (and v_2 white), and let $S_1' = S_1 - \{v_1\}$ and $S_2' = S_2 - \{v_2\}$. Then S_1' and S_2' are both Temperleyan polyominoes, and tileable. And $\{v_1, v_2\}$ is tileable by a single domino since v_1 and v_2 are neighbors. The three regions form a partition of R, and are all tileable, and therefor R is tileable (Theorem 2).*

Theorem 129. *The gap number of an even polyomino with H holes is $|H - 1|$.*

[Not referenced]

Proof. *Case $H = 0$.* If we remove a black corner cell, the resulting polyomino is Temperleyan and thus tileable by Theorem 128. Therefor, the gap number is 1.

Case $H = 1$. The region is tileable (Theorem 126), and so the gap number is 0.

Case $H > 1$. Form a Temperleyan polyomino R' by appending black cells v_i to all but one interior border; there are $H - 1$ such cells. This polyomino is tileable (Theorem 128). Let u_i be the neighbor of v_i in the same domino of a tiling of R', and let $R'' = R - \{u_i\}$. Then R'' is tileable, and R therefor has a tiling by dominoes and $H - 1$ monominoes. Since $g \geq |\Delta(R)|$ (Theorem 108), and $\Delta(R) = 1 - H$ (Theorem 125), the tiling is optimal, and therefor $g = |1 - H|$. □

4.4 *Summary of tiling criteria*

4.5 *Miscellaneous*

4.5.1 *Polyominoes that can be Tiled by Dominoes*

How many regions of a given area is tileable by dominoes? Table 14 gives us a sense. A polyomino tileable by dominoes is called a **polydomino**.

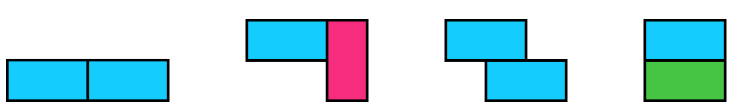


Figure 130: The four tileable tetrominoes.

Conditions for tileability	Theorem
Necessary	
$\lvert R \rvert$ must be even.	1
R is balanced.	23
The parity of dominoes that cross the border of any subregion S of R must equal the parity of the deficiency of S.	21
The deficiency of any subregion must match the flow (and therefor, the number border of S must be able to accommodate the necessary flow.	25
R must be have least 2 sides of even length.	29
$R \ominus S$ is tileable if S is an n-cylinder with n even, and removing S from R is safe.	38
R is fair by some discriminating coloring.	47
Sufficient	
All sides of R is even.	30
$R \ominus S$ is tileable, with S a n-cylinder with n even	37
R is a strip-polyomino with even area, which includes rectangles with even area.	51
R is saturnian	55
R is a checkerboard-colored strip polyomino with two cells of opposite colors removed	56
R is a balanced stack polyomino (which includes balanced Young diagrams).	46 (41)
R is a balanced jig-saw region.	45
R is an even polyomino with one hole.	126
Necessary and Sufficient	
It has no bad patches	34
R is fair by all discriminating colorings	49
We can consistently assign heights in each step of Thurston's Algorithm.	92
R is simply connected, balanced, and the difference in height is smaller than the distance between any two vertices on the border.	95

Table 13: Summary of tiling criteria for domino tilings.

n	2n-ominoes A210996	Balanced A234012	Tileable A056785	Uniquely tileable A213377
1	1	1	1	1
2	5	4	4	3
3	35	24	23	20
4	369	230	211	170
5	4655	2601	2227	1728
6	63600	32810	25824	18878
7	901971	433855	310242	214278
8	13079255	5923677	3818983	2488176
9	192622052		47752136	29356463
α	14.7*	13.6*	12.5*	11.8*

Table 14: Balanced, tileable and uniquely tileable dominoes.

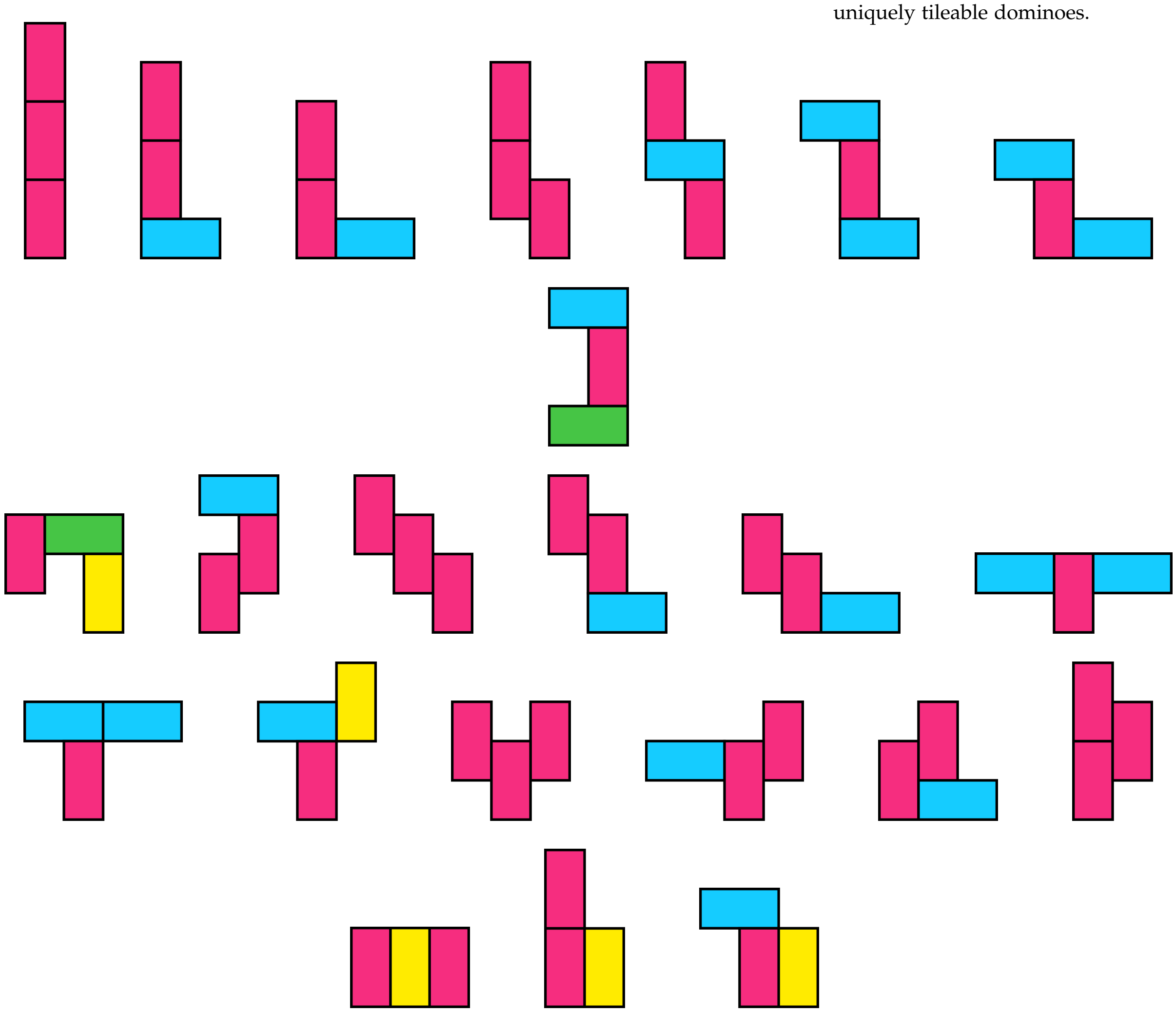


Figure 131: The 23 tileable hexominoes. All except the last three have unique tilings.

4.6 Further Reading

Berlov and Kokhas (2004) give some interesting problems (with solutions) on domino tilings.

For more on orders and lattices, see for example Davey and Priestley (2002) and Roman (2008). The following properties of lattices are well-known and it is worth considering their implication on domino tilings:

(1) M_3 is not isomorphic to any sublattice. (Davey and Priestley, 2002, Thereom 4.10, p. 89)

(2) N_5 is not isomorphic to any sublattice. (Davey and Priestley, 2002, Thereom 4.10, p. 89)

(3) The covering relation $\prec$ forms a median graph. (Roman, 2008, Ex. 9, p. 125)

For more on the lattice structure of domino tilings specifically, see Rémila (2004), Caspard et al. (2003). For slightly more general results on height functions and dominoes tilings see Ito (1996), Beauquier and Fournier (2002) and Bodini and Latapy (2003).

Domino tilings (especially the use of height functions and the structure of tilings) have also been considered for regions more general than was covered here. For example:

- plane regions with barriers (Propp and Stanley, 1999)
- three-dimensional regions (Milet, 2015)
- regions on a torus (Impellizieri, 2016)
- quadriculated surfaces (Saldanha et al. (1995) and Saldanha and Tomei (2003))

Height functions and lattices can also be used to study other tiling problems[11], for example, tilings with:

[11] I use here the convenient list provided by Kenyon.

- bars (Kenyon and Kenyon, 1992)
- rectangles (Kenyon and Kenyon, 1992)
- squares (Kenyon, 1993)
- tetrominoes (Muchnik and Pak, 1999)
- bibones (Kenyon and Rémila, 1996)
- tribones (Thurston, 1990)
- lozenges (Rémila (1996) and Thurston (1990))
- leaning dominoes (Rémila, 1996)
- triangles (Rémila, 1996)

In our treatment of height functions, we did not deal with regions with holes. The best explanation for dealing with holes is Bodini and Fernique (2006), but it is covered in the slightly more general setting of *planar dimer tilings*. Desreux et al. (2004) gives a similar technique specific for domino tilings, including algorithms for finding minimum tilings, to generate all tilings, and to do uniform sampling of tilings. Saldanha et al. (1995) gives an alternative approach, but requires some ideas from topology. Thiant (2003) gives an algorithm for tiling any region with dominoes.

In many of the proofs I omitted the details of solving recurrence relations. How to solve these is explained in many books on combinatorics or discrete mathematics; see for example (Brualdi, 2009, Chapter 7), (Knuth et al., 1989, Chapters 1 and 7) and (Wilf, 2005, Chapter 1).

Propp and Lowell (2015) discusses enumeration of tilings in general, and is a good overview of some of the techniques used. Mathematicians calculated the number of domino tilings of various classes, for example:

- quartered Aztec diamond (Lai, 2014)
- deficient Aztec rectangle (Krattenthaler, 2000)
- holy square (Tauraso, 2004)
- L-shaped domain (Colomo et al., 2018)
- expanded Aztec diamond Oh (2018)

Counting more general types of matchings is discussed in Propp et al. (1999). Special attention has been given to matchings whose counts are perfect powers (such as the Aztec diamond). See for example Pachter (1997), Ciucu (2003) and Lai (2013). One notable result is a factorization theorem for graphs with reflective symmetry given in Ciucu (1997).

Numerical results and generating functions for various tilings of rectangles by small rectangles (including dominoes) is given in Mathar (2013) and Mathar (2014). This includes counts of tatami tilings. In addition to generating functions, Gershon (2015) gives exact formulas for rectangles of fixed width up to 6.

For more details on tatami tilings with dominoes only, see Alhazov et al. (2010) and Ruskey and Woodcock (2009). For more on tatami tilings with dominoes and monominoes, see Erickson (2013), Erickson et al. (2011) and Erickson and Schurch (2012). The idea of *water striders* in Erickson et al. (2011) is closely related to Theorems 98, 97, and 96.

We consider optimal tilings of rectangles by rectangles and the associated gap number in Section 6.1.2. Fault-free tilings of other polyominoes is covered in Section 6.4. Temperleyan dominoes are closely related to spanning trees of certain graphs. This is discussed in Kenyon et al. (2001), with applications to the dimer model in Kenyon (2000b), Kenyon (2001) and Kenyon (2000a).

5
Various Topics

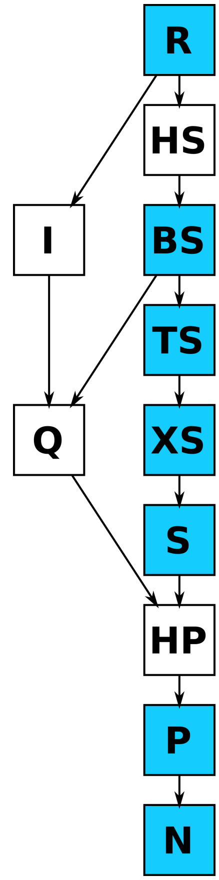


Figure 132: The relationship between classes. Classes in blue blocks have elements that are not in any higher class; white ones have no examples of polyominoes that are not also in a higher class.

The sections in this chapter can be considered “short chapters”; they are independent topics that introduce material useful for later chapters, but without enough material to warrant their own chapters.

5.1 *The Tiling Hierarchy*

Up to this point, we have looked mostly of tilings of arbitrary regions. In this section, we look at tilings of some “fundemental” regions, most of which are not finite. We will also see that the regions considered in this section forms a hierarchy; if a polyomino set tiles one of these regions, it tiles all the regions below it in the hierarchy.

These regions form a hierarchy (as established in Theorem 133), so that when we know something about whether a polyomino tiles a certain class, we also have information about whether it can tile some of the other classes.

The tiling hierarchy is the classes defined in Golomb (1966) and extended in Liu (2018a). We also split the original Plane class into two (aperiodic and periodic).

A **reptile** is a polyomino that can tile a scaled copy of itself.

Monominoes tile everything, including bigger squares, therefor monominoes are reptiles. So is any rectangle. The first interesting reptile is the straight tromino. The skew tetromino is the smallest polyomino that is not a reptile. The T-tetromino is a reptile — but in a boring way: it tiles a square, and so we can use the square to tile a bigger T-tetromino. Of course this is a general idea: if a polyomino tiles a rectangle, it can tile a square, and if it tiles a square, it can tile any scaled polyomino, including itself. Although the converse is not proven to be true, it is an open problem (Winslow, 2018, Open Problem 13)]; all reptiles we know also tile rectangles.

That reptiles tile the plane should be clear: take the polyomino, dissect it in smaller polyominoes of the same shape. Scale the figure so that the polyominoes in the dissection are the same size as the

Class	Can Tile
N	Nothing
AP	Plane (aperiodic)
PP	Plane (periodic), torus
HP	Half plane
Q	Quadrant
S	Strip, Cylinder
XS	Crossed Strip
TS	Branched Strip
HS	Half-strip
BS	Bent-strip
R	Rectangle
I	Scaled copy of itself

Table 15: Tiling Classes. For example, if some polyomino $P \in \mathbf{R}$ it means it can tile a rectangle.

original. Repeat. This way, any size patch can be covered. (This type of tiling is called a **substitution tiling**.)

What is perhaps surprising is that reptiles always tile a quadrant. We prove this in the next few theorems.

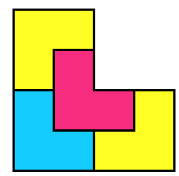


Figure 133: The T-tetromino is a reptile.

Theorem 130 (Golomb (1966), Theorem 4)**.** *A reptile covers at least one corner of its rectangular hull.*

[Referenced on page 167]

Proof. Make a big version of the reptile by the replication process, (as big as you want). Clearly, if the reptile does not cover all four corners, there is no why it can cover the corner of the bigger copy; there will always be a gap. □

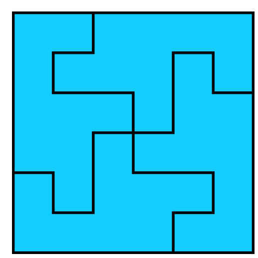


Figure 134: An example of a reptile that covers only one corner of its rectangular hull.

A reptile can fail to cover two corners; 134 shows an example.

Theorem 131. *A polyomino that can tile a rectangle, can also tile a square.*

[Referenced on page 125]

Proof. Suppose the polyomino tiles $R(m,n)$. You can stack n of these rectangles together to tile $R(mn,n)$, and then stack m of *these* to tile $R(mn,mn)$, which is a square. □

Theorem 132. *A polyomino that tiles a rectangle tiles can tile any polyomino scaled by a factor.*

[Not referenced]

Proof. By Theorem 131 a polyomino that tiles a rectangle can tile a square. These squares can be stacked to construct the scaled polyomino. □

The constructions in the proofs of the two theorems above does not necessarily construct the smallest possibility. For example, a 2×4 polyomino can also tile a 4×4 square.

Theorem 133 (Golomb (1966))**.** *For a single polyomino:*

$$\mathsf{BS} \subset \mathsf{Q} \subset \mathsf{HP} \tag{5.1}$$

$$\mathsf{R} \subset \mathsf{I} \subset \mathsf{Q} \subset \mathsf{HP} \tag{5.2}$$

$$\mathsf{R} \subset \mathsf{HS} \subset \mathsf{BS} \subset \mathsf{TS} \subset \mathsf{XS} \subset \mathsf{S} \subset \mathsf{HP} \subset \mathsf{PP} \subset \mathsf{N} \tag{5.3}$$

[Referenced on pages 124 and 193]

Proof. All the inclusions are obvious except for the following two:

$$\mathsf{XS} \subset \mathsf{S} \tag{5.4}$$

$$\mathsf{I} \subset \mathsf{Q} \tag{5.5}$$

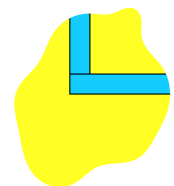
To prove 5.4, let P tile a crossed strip. Let m be the maximum dimension of the polyomino's rectangular hull, and let w be the width of one arm of the crossed strip. Now in any $w \times m$ rectangle of that arm, we can draw at least one path from the top of the arm to the bottom. There are only finitely many of such patterns, so in the arm with infinitely many such rectangles, at least one pattern must repeat. The segment of the arm between such repeated patterns is a cylinder, and it can used to tile a strip by placing them end to end.

To prove 5.5, let P be a reptile. It must cover a corner of its rectangular hull (by Theorem). Place the polyomino in the first quadrant with this corner at the origin. Now perform the replication process until the covered corner of the big polyomino is covered by a small polyomino in the same orientation. (This process can go on for at most 8 iterations, since there are only 8 distinct orientations .) Move this bigger polyomino to the origin, and call the new figure $F(P)$. By repeatedly applying F, we can expand the tiling, which will cover the entire quadrant in the limit. □

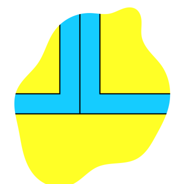
While we often use this theorem in the "forward" direction, it is worth considering reverse implications. For example, if a polyomino does not tile a quadrant, it can also not tile a rectangle. The following example shows how knowledge about how a tile tiles the quadrant informs us about how it tiles a rectangle.

Example 15 (Reid (2003), Example 6.7)**.** *The polyomino* $B(2 \cdot 3 \cdot 2^2)$ *tiles* $R(m, n)$ *iff* $6 \mid m$ *and* $6 \mid n$*.*

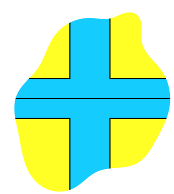
Proof. First, note that of all 8 possible placements along an edge, only 4 are possible. Second, by examining all cases (not hard to do) we observe that the edge must always be made of pairs as shown below.



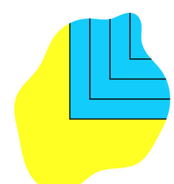
(a) Half strips tile a bent strip.



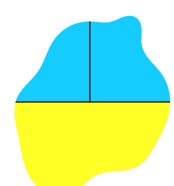
(b) Bent strips tile a branched strip.



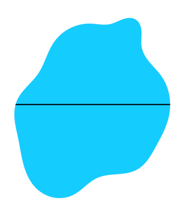
(c) Branched strips tile a crossed strip.



(d) Bent strips tile a quadrant.



(e) Quadrants tile a half plane.



(f) Half planes tile the plane.

Figure 135: Some inclusions of the tiling hierarchy demonstrated.

Further examination shows the only way to tile the quadrant is to use 6×6 squares. And since no other tiling of the quadrant exists, it implies all rectangles must be tiles using these squares too, which is only possible if $6 \mid m$ and $6 \mid n$. $\square$

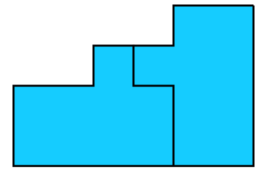


Figure 136: These polyominoes can only pack an edge in the configuration shown.

No-one knows whether the following statements are true (Winslow (2018, Open Problems 12, 13), Golomb (1996, p. 107–108).)

(1) $\mathbf{I} = \mathbf{R}$

(2) $\mathbf{HS} = \mathbf{R}$

(3) $\mathbf{Q} = \mathbf{BS}$ (and indeed, if $\mathbf{I} \neq \mathbf{R}$, then whether $\mathbf{Q} = \mathbf{I} \cup \mathbf{BS}$).

(4) $\mathbf{HP} = \mathbf{S}$

(5) $\mathbf{PP} = \mathbf{AP}$

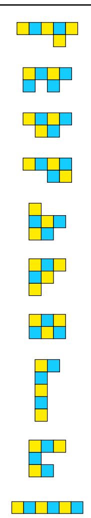
There is no reason to believe any of these statements, other than that no counter examples have been found. There are suggestions that these may not be true:

- The statements (2)-(5) do not hold for tilesets that have more than one polyomino. For example, .
- The Y pentomino tile half-strips of width 6 and 8, bit not any rectangles with sides of length 6 or 8. This suggests $\mathbf{HS} \neq \mathbf{R}$.
- The P pentomino is an example of a reptile that does not require construction of a rectangle first.

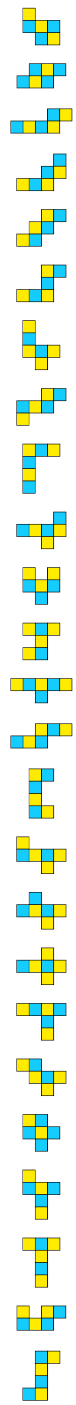
All polyominoes with area 6 or less can (at least) tile the plane. Table 16 lists the highest classes that polyominoes belong to.

Problem 36. *Is the following true or false: If a polyomino tiles another polyomino with a scaling factor of $n > k$, where k is the largest side of the rectangular hull of the polyomino, it tiles a rectangle.*

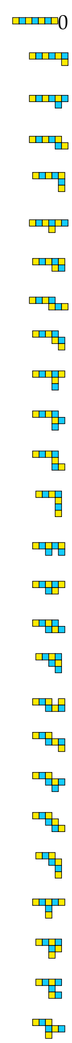
Problem 37. *Complete the Table 16.*



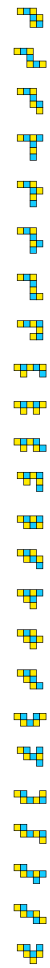
Polyomino	Class
Monomino	R
Domino	R
Bar	R
Right	R
Bar	R
Square	R
T	R
L	R
Skew	BS
I	R
L	R
P	R
Y	R
N	BS
W	BS
F	TS
V	TS
T	PP
U	PP
X	PP
Z	PP
	R
	R
	R
	R
	R
	R
	R
	R
	R
	R



Polyomino	Class
	BS
	BS
	BS
	BS
	BS
	BS
	TS
	TS
	TS
	XS
	XS
	XS
	S
	S
	S
	PP
	PP
	PP
	PP
	PP
	PP
	PP
	PP
	PP
	PP



Polyomino	Class
	R
	PP
	PP*, Q′
	PP
	PP*, Q′
	PP*
	R
	PP
	N*
	S*
	R
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*
	N*



Polyomino	Class
	N*
	N*
	N*
	N*

Table 16: Tiling classes for polyominoes. The table gives the smallest class a polyomino is part of. Classes marked with an asterisk is not guaranteed to be the smallest. An accent shows the polyomino is *not* in that class.

5.1.1 *Strips, Crossed Strips, Bent Strips, and Half Strips*

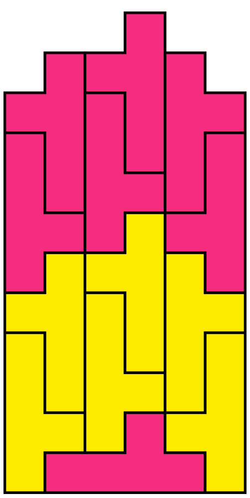
We make the following set of definitions:

- A strip is **prime** if no smaller strip tiles it.
- A half strip is prime if no smaller half strip tiles it.
- A bent strip is prime if no smaller bent strip tiles it.
- A rectangle is prime if no smaller rectangle tiles it.

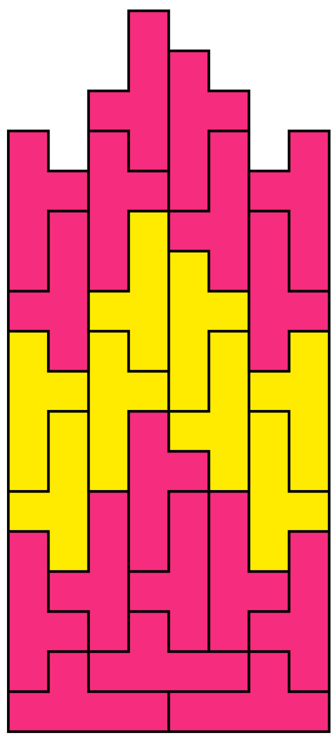
Similar definitions hold for larger tile sets.

From prime regions of a tile set, we can construct all regions of the same type that is tileable by the set. We can also, to some extent, given an arbitrary region, answer whether it is tileable by a tile set if we know all the prime region of the relevant type. The case for rectangles is interesting, and treated in detail in the chapter Rectangles.

For strips and half strips, finding all strips or half strips is equivalent to the Frobenius problem.The prime strips and halfstrips are given in table 17. A rectangle $R(m,n)$ can be stacked to form a strip of widths m or n, but not all strips correspond to prime rectangles. Those that don't are marked in the table.



(a) HS(6)



(b) HS(8)

Figure 137: Halfstrips of the Y pentomino that does not correspond to widths of prime rectangles.

Theorem 134. *If a tile set tiles a strip, there exists a periodic tiling by the same set.*

Polyomino	Prime Half Strips	Prime Strips
Monomino	1	1
Domino	1	1
Bar	1	
Right	2 3	2 3
Bar	1	1
Square	2	2
L	2 3	2 3
T	4,...?	2, 13[d]
F	-	-
I	1	1
L	2 5	2 5
N	-	-
P	2 3	2 3
T	-	-
U	-	-
V	-	5 ...?
W	5 ...?	
X	-	-
Y	5, 6[a], 8[a], 9	2[b], 5
Z	-	-
Y[c]	12[a], 23, 29, 30, 32	2, ...?

Table 17: Prime Half Strips and Strips.
[a]No rectangles with this width.
[b]No half strips with this width.
[c]Prime ?
[d]Hochberg (2015)

[Not referenced]

Proof. Lets suppose we have a tiling of a strip by a set of polyominoes.

Now make new tiles by taking each column as a tile. Tiles are the same if they cut though the original polyominoes in exactly the same way. This new tile set is finite. Now construct a directed graph with the new tiles as vertices, with an edge from T_i to T_j if T_j can be placed on the right of T_i and is legal by considering the original set (that is, all the polyominoes formed are from the original set).

The tiling if the strip corresponds to an infinite path along the directed edges. But since there is a finite number of vertices, there must be a cycle in the graph. This cycle corresponds to a periodic tiling with the tiles, and so also a periodic tiling with the original tiles. □

A cycle in the proof above corresponds to a cylinder. The theorem is then equivalent to the following:

Theorem 135. *If a tile tiles a strip of width m, it also tiles a cylinder of width m.*

[Not referenced]

Problem 38. *Complete Table 17.*

5.1.2 *Extending the hierarchy*

It is not hard to come up with various other classes that fits into the existing hierarchy. It is much harder to tell whether such extensions would be useful. In this section, I give some ideas for classes that might be.

RR Tiles a rectangle with a rectangular hole. It is easy to see that **R** ⊂ **RR**. The T- and V-pentominoes are examples of polyominoes in **RR** but not in **R**.

Rectangle with n rectangular holes

Rectangle with an arbitrary connected hole

Rectangle with n arbitrary holes

Corner

Edge

Half Edge

Open Edge

The plane with holes

The more classes there are, the more tedious it is to find the characteristic class of a polyomino or set of polyominoes. In the table below we only classify polyominoes up to pentominoes.

The notion of *prime shape* also becomes more complicated, although it may be interesting to explore.

5.1.3 Further Reading

Reid (1997) give several examples of families of polyominoes that tile strips, but it is unknown whether they tile any rectangles.

Nitica (2018a) and Nitica (2018b) explores how the tiling hierarchy is affected when we are only allowed to tile by translation. Other classes of regions is also considered.

5.2 Colorings

If we color the plane with k colors, how many colors do we need so that a polyomino does not cover two cells of the same color no matter how it is placed?

If we need k colors for a polyomino P, we call this the **chromatic number** of P and denote it by $C(P)$. We will use $C(m,n)$ as an abbreviation for $C(R(m,n))$. A coloring of the plane with $C(P)$ colors such that however we place P, it does not cover two cells of the same color is called an **sufficient coloring** for P. A sufficient colorings that minimize the number of colors — and therefor use k colors — is called an **optimal coloring** for P;

We use the following notation for some common colorings. The first is a generalization of the checkerboard coloring, which we call a **flag coloring**.

$$F_{m,n}(x,y) = (x + yn) \bmod m. \tag{5.6}$$

In this coloring, there are m colors, each repeated in every row (in the same order), and each row offset by n cells from the previous row.

If $n = 1$, we simply write F_m. In this notation, the checkerboard coloring is F_2.

The second is called a **square coloring**.

$$S_m(x,y) = (x \bmod m) + (y \bmod m)m. \tag{5.7}$$

And this is simply the grid divided into $m \times m$ squares, each containing m^2 colors in the same pattern.

The final type of coloring is their product:

$$S_k \times F_{m,n}(x, y) = F_{m,n}(\lfloor x/k \rfloor, \lfloor y/k \rfloor)k^2 + S_k(x, y). \tag{5.8}$$

Problem 39. *Show that the number of colors for*

(1) *$F_{m,n}$ is n,*

(2) *S_m is m^2, and*

(3) *$S_k \times F_{m,n}$ is km.*

Theorem 136 (Yu (2014))**.** *For any polyomino P, $C(P) \geq |P|$.*

[Referenced on page 134]

Proof. Obvious. □

Polyominoes for which $C(P) = |P|$ are called **efficient**. Otherwise they are **inefficient**.

Theorem 137 (Yu (2014), Implicit)**.** *Suppose the hull of P is $R(m, n)$. Then $C(P) \leq C(R(m, n))$.*

[Referenced on pages 134 and 141]

Proof. Suppose $C(P) > C(m, n)$. Color the plane with an optimal coloring for $R(m, n)$. Since this coloring requires less than $C(P)$ colors, some placement of P covers two cells of the same color. If we put $R(m, n)$ to cover P in such a placement, then two covered by $R(m, n)$ must have the same color, and therefor the coloring cannot be optimal for $R(m, n)$, a contradiction. Therefor $C(P) \leq C(m, n)$. □

If P can cover any two cells of a polyomino Q, we say P **protects** Q. A polyomino that cannot protect a polyomino larger than itself is called **weak**. Bars and squares are weak.

Problem 40. *Which rectangles are weak?*

Theorem 138. *All efficient polyominoes are weak.*

[Not referenced]

The converse does not hold. For example, below we will see that W-polyominoes with an odd number of cells are also weak, but they are not efficient.

Theorem 139 (Yu (2014), Implicit)**.** *If P protects Q, then $C(P) \geq C(Q)$.*

[Referenced on pages 134 and 141]

Proof. Suppose $C(Q) > C(P)$. Color the plane with an optimal coloring for P. This coloring requires less than $C(Q)$ colors, some placement of Q covers two cells of the same color. These cells can be covered by P, so the coloring cannot be optimal for P, a contradiction. Therefor, $C(Q) \leq C(P)$. □

Theorem 140 (Yu (2014), Implicit). *If $Q \subset P$ then $C(P) \geq C(Q)$.*

[Referenced on pages 139 and 141]

Proof. P can cover the whole of Q, and in particular any two cells of Q. Therefor, by Theorem 139 $C(P) \geq C(Q)$. □

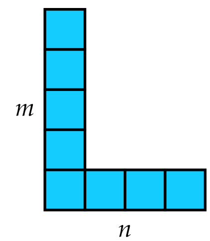


Figure 138: The shape of Theorem 141: $B(1^m \cdot (n-1))$.

Theorem 141. *Let $P = B(1^m \cdot (n-1))$. Then $C(P) = C(m,n)$, with $F_{m(m+1)-1,m(m+2)}$ a sufficient coloring.*

[Referenced on pages 134 and 141]

Proof. The hull of P is $R(m,n)$, so $C(P) \leq C(m,n)$ (Theorem 137).

P can cover any two cells of $R(m,n)$, so $C(P) \geq C(m,n)$ (Theorem 139).

Taking these together, we get $C(P) = C(m,n)$. □

Theorem 142. *Suppose that P has a optimal coloring with n colors, and the coloring is a square coloring. Then if we remove a cell from P and append it so that the removed cell has the same color to form Q, then $C(Q) = C(P)$.*

[Not referenced]

Problem 41. *What are all the polyominoes which are equivalent under S_3 to the A-hexomino ($B(1 \cdot 2 \cdot 3)$)?*

Theorem 143. *If P fits into Q in all orientations, and Q tiles the plane by translation*[1]*, then $C(P) \leq |Q|$.*

[Referenced on pages 134 and 141]

Theorem 144. *If $P \in$ **All** tiles the plane, then $C(P) = |P|$, and so P is efficient.*

[Referenced on pages 134 and 141]

Proof. P fits in itself in all orientations, so by Theorem 143 we have $C(P) \leq P$. But by Theorem 136, we have $C(P) \geq P$. Thus, $C(P) = |P|$. □

Theorem 145. *uppose the period of the tilings of Theorems 143 or 144 is given by $u = (m,0)$ and $v = (n,1)$, then*

[1] A polyomino that tiles the plane by translation is called an *exact polyomino*; exact polyominoes are discussed in Section 7.1.

[Not referenced]

The next few Theorems establishes what is known of $R(m,n)$, an important case since the rectangular hull establishes an upper bound for the chromatic number of all polyominoes. One would imagine calculating the chromatic number of rectangles is easy; however, the problem is subtle and not completely solved.

	n																			
m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	4	8	8	12	12	16	16	20	20	24	24	28	28	32	32	36	36	40	40
3	3	8	9	15	18	18	24	27	27	33	36	36	42	45	45	51	54	54	60	63
4	4	8	15	16	24	32	32	32	40	48	48	48	56	64	64	64	72	80	80	80
5	5	12	18	24	25	35	45	50	50	50	60	70	75	75	75	85	95	100	100	100
6	6	12	18	32	35	36	48	60	72	72	72	72	84	96	108	108	108	108	120	132
7	7	16	24	32	45	48	49	63	77	91	98	98	98	98	112	126	140	147	147	147
8	8	16	27	32	50	60	63	64	80	96	112	128	128	128	128	128	144	160	176	192
9	9	20	27	40	50	72	77	80	81	99	117	135	153	162	162	162	162	162	180	198
10	10	20	33	48	50	72	91	96	99	100	120	140	160	180	200	200	200	200	200	200
11	11	24	36	48	60	72	98	112	117	120	121	143	165	187	209	231	242	242	242	242
12	12	24	36	48	70	72	98	128	135	140	143	144	168	192	216	240	264	288	288	288
13	13	28	42	56	75	84	98	128	153	160	165	168	169	195	221	247	273	299	325	338
14	14	28	45	64	75	96	98	128	162	180	187	192	195	196	224	252	280	308	336	364
15	15	32	45	64	75	108	112	128	162	200	209	216	221	224	225	255	285	315	345	375
16	16	32	51	64	85	108	126	128	162	200	231	240	247	252	255	256	288	320	352	384
17	17	36	54	72	95	108	140	144	162	200	242	264	273	280	285	288	289	323	357	391
18	18	36	54	80	100	108	147	160	162	200	242	288	299	308	315	320	323	324	360	396
19	19	40	60	80	100	120	147	176	180	200	242	288	325	336	345	352	357	360	361	399
20	20	40	63	80	100	132	147	192	198	200	242	288	338	364	375	384	391	396	399	400

Table 18: Computed values of $C(m,n)$, assuming that colorings are periodic. Colored cells are values that have been proven:

- Theorem 147
- Theorem 148
- Theorem 152

Table 18 shows some values computed by computer. Only those marked are proved to be the lowest; I calculated the remainder of the values by exhaustively searching for the smallest periodic coloring. It is not established that there exist an optimal coloring that is periodic.

Theorem 146. *Assume $m \le n$. Then*

$C(m,n) \le m(2n-m)$.

[Not referenced]

Proof. All orientations of $R(m,n)$ fit in $B(n^m, m^{n-m})$. Therefor $C(m,n) \le |B(n^m, n^{n-m})| = nm + m(n-m) = m(2n-m)$. ☐

Theorem 147 (Puleo (2018a))**.** *Suppose $m \le n$. Then $C(m,n) = mn$ if and only if m divides n, and so*

(1) *$C(m,1) = m$, with coloring F_m*

(2) *$C(m,m) = m^2$, with coloring S_m.*

[Referenced on pages 135, 139 and 141]

	2	3	4	5	6	7	8	9	10
2	S_2	$F_{8,3}$	$S_2 \times F_2$	$F_{12,5}$	$S_2 \times F_3$	$F_{16,7}$	$S_2 \times F_4$	$F_{20,9}$	$S_2 \times F_5$
3	$F_{8,3}$	S_3	$F_{15,4}$	$S_3 \times F_2$	$S_3 \times F_2$	$F_{24,7}$	$S_3 \times F_3$	$S_3 \times F_3$	$F_{33,10}$
4	$S_2 \times F_2$	$F_{15,4}$	S_4	$F_{24,5}$	$S_6 \times F_{8,3}$	$S_4 \times F_2$	$S_4 \times F_2$	$F_{40,9}$	$S_{10} \times F_{12,5}$
5	$F_{12,5}$	$S_3 \times F_2$	$F_{24,5}$	S_5	$F_{35,6}$	$F_{45,19}$	$S_5 \times F_2$	$S_5 \times F_2$	$S_5 \times F_2$
6	$S_2 \times F_3$	$S_3 \times F_2$	$S_6 \times F_{8,3}$	$F_{35,6}$	S_6	$F_{48,7}$	$S_8 \times F_{15,4}$	$S_9 \times F_{8,3}$	$S_6 \times F_2$
7	$F_{16,7}$	$F_{24,7}$	$S_4 \times F_2$	$F_{45,19}$	$F_{48,7}$	S_7	$F_{63,8}$	$F_{77,34}$	$F_{91,27}$
8	$S_2 \times F_4$	$S_3 \times F_3$	$S_4 \times F_2$	$S_5 \times F_2$	$S_8 \times F_{15,4}$	$F_{63,8}$	S_8	$F_{80,9}$	$S_{10} \times F_{24,5}$
9	$F_{20,9}$	$S_3 \times F_3$	$F_{40,9}$	$S_5 \times F_2$	$S_9 \times F_{8,3}$	$F_{77,34}$	$F_{80,9}$	S_9	$F_{99,10}$
10	$S_2 \times F_5$	$F_{33,10}$	$S_{10} \times F_{12,5}$	$S_5 \times F_2$	$S_6 \times F_2$	$F_{91,27}$	$S_{10} \times F_{24,5}$	$F_{99,10}$	S_{10}
11	$F_{24,11}$	$S_3 \times F_4$	$S_4 \times F_3$	$F_{60,11}$	$S_6 \times F_2$	$S_7 \times F_2$	$F_{112,41}$	$F_{117,53}$	$F_{120,11}$
12	$S_2 \times F_6$	$S_3 \times F_4$	$S_4 \times F_3$	$F_{70,29}$	$S_6 \times F_2$	$S_7 \times F_2$	$S_{12} \times F_{8,3}$	$S_{12} \times F_{15,4}$	$S_{12} \times F_{35,6}$
13	$F_{28,13}$	$F_{42,13}$	$F_{56,13}$	$S_5 \times F_3$	$F_{84,13}$	$S_7 \times F_2$	$S_8 \times F_2$	$F_{153,35}$	$F_{160,49}$
14	$S_2 \times F_7$	$S_3 \times F_5$	$S_{14} \times F_{16,7}$	$S_5 \times F_3$	$S_{14} \times F_{24,7}$	$S_7 \times F_2$	$S_8 \times F_2$	$S_9 \times F_2$	$S_38 \times F_{45,19}$
15	$F_{32,15}$	$S_3 \times F_5$	$S_4 \times F_4$	$S_5 \times F_3$	$S_{15} \times F_{12,5}$	$F_{112,15}$	$S_8 \times F_2$	$S_9 \times F_2$	$S_{15} \times F_{8,3}$
16	$S_2 \times F_8$	$F_{51,16}$	$S_4 \times F_4$	$F_{85,16}$	$S_6 \times F_3$	$F_{126,55}$	$S_8 \times F_2$	$S_9 \times F_2$	$S_{10} \times F_2$
17	$F_{36,17}$	$S_3 \times F_6$	$F_{72,17}$	$F_{95,39}$	$S_6 \times F_3$	$F_{140,41}$	$F_{144,17}$	$S_9 \times F_2$	$S_{10} \times F_2$
18	$S_2 \times F_9$	$S_3 \times F_6$	$S_{18} \times F_{20,9}$	$S_5 \times F_4$	$S_6 \times F_3$	$S_7 \times F_3$	$S_{18} \times F_{40,9}$	$S_9 \times F_2$	$S_{10} \times F_2$
19	$F_{40,19}$	$F_{60,19}$	$S_4 \times F_5$	$S_5 \times F_4$	$F_{120,19}$	$S_7 \times F_3$	$F_{176,65}$	$F_{180,19}$	$S_{10} \times F_2$
20	$S_2 \times F_10$	$S_3 \times F_7$	$S_4 \times F_5$	$S_5 \times F_4$	$S_{20} \times F_{33,10}$	$S_7 \times F_3$	$S_{20} \times F_{12,5}$	$F_{198,89}$	$S_{10} \times F_2$

	11	12	13	14	15	16	17	18	19	20
2	$F_{24,11}$	$S_2 \times F_6$	$F_{28,13}$	$S_2 \times F_7$	$F_{32,15}$	$S_2 \times F_8$	$F_{36,17}$	$S_2 \times F_9$	$F_{40,19}$	$S_2 \times F_10$
3	$S_3 \times F_4$	$S_3 \times F_4$	$F_{42,13}$	$S_3 \times F_5$	$S_3 \times F_5$	$F_{51,16}$	$S_3 \times F_6$	$S_3 \times F_6$	$F_{60,19}$	$S_3 \times F_7$
4	$S_4 \times F_3$	$S_4 \times F_3$	$F_{56,13}$	$S_{14} \times F_{16,7}$	$S_4 \times F_4$	$S_4 \times F_4$	$F_{72,17}$	$S_{18} \times F_{20,9}$	$S_4 \times F_5$	$S_4 \times F_5$
5	$F_{60,11}$	$F_{70,29}$	$S_5 \times F_3$	$S_5 \times F_3$	$S_5 \times F_3$	$F_{85,16}$	$F_{95,39}$	$S_5 \times F_4$	$S_5 \times F_4$	$S_5 \times F_4$
6	$S_6 \times F_2$	$S_6 \times F_2$	$F_{84,13}$	$S_{14} \times F_{24,7}$	$S_{15} \times F_{12,5}$	$S_6 \times F_3$	$S_6 \times F_3$	$S_6 \times F_3$	$F_{120,19}$	$S_{20} \times F_{33,10}$
7	$S_7 \times F_2$	$S_7 \times F_2$	$S_7 \times F_2$	$S_7 \times F_2$	$F_{112,15}$	$F_{126,55}$	$F_{140,41}$	$S_7 \times F_3$	$S_7 \times F_3$	$S_7 \times F_3$
8	$F_{112,41}$	$S_{12} \times F_{8,3}$	$S_8 \times F_2$	$S_8 \times F_2$	$S_8 \times F_2$	$S_8 \times F_2$	$F_{144,17}$	$S_{18} \times F_{40,9}$	$F_{176,65}$	$S_{20} \times F_{12,5}$
9	$F_{117,53}$	$S_{12} \times F_{15,4}$	$F_{153,35}$	$S_9 \times F_2$	$S_9 \times F_2$	$S_9 \times F_2$	$S_9 \times F_2$	$S_9 \times F_2$	$F_{180,19}$	$F_{198,89}$
10	$F_{120,11}$	$S_{12} \times F_{35,6}$	$F_{160,49}$	$S_38 \times F_{45,19}$	$S_{15} \times F_{8,3}$	$S_{10} \times F_2$	$S_{10} \times F_2$	$S_{10} \times F_2$	$S_{10} \times F_2$	$S_{10} \times F_2$
11	S_{11}	$F_{143,12}$	$F_{165,76}$	$F_{187,67}$	$F_{209,56}$	$F_{231,43}$	$S_{11} \times F_2$	$S_{11} \times F_2$	$S_{11} \times F_2$	$S_{11} \times F_2$
12	$F_{143,12}$	S_{12}	$F_{168,13}$	$S_{14} \times F_{48,7}$	$S_{15} \times F_{24,5}$	$S_{16} \times F_{15,4}$	$F_{264,109}$	$S_{18} \times F_{8,3}$	$S_{12} \times F_2$	$S_{12} \times F_2$
13	$F_{165,76}$	$F_{168,13}$	S_{13}	$F_{195,14}$	$F_{221,103}$	$F_{247,77}$	$F_{273,64}$	$F_{299,116}$	$F_{325,51}$	$S_{13} \times F_2$
14	$F_{187,67}$	$S_{14} \times F_{48,7}$	$F_{195,14}$	S_{14}	$F_{224,15}$	$S_{16} \times F_{63,8}$	$F_{280,99}$	$S_68 \times F_{77,34}$	$F_{336,71}$	$S_54 \times F_{91,27}$
15	$F_{209,56}$	$S_{15} \times F_{24,5}$	$F_{221,103}$	$F_{224,15}$	S_{15}	$F_{255,16}$	$F_{285,134}$	$S_{18} \times F_{35,6}$	$F_{345,91}$	$S_{20} \times F_{15,4}$
16	$F_{231,43}$	$S_{16} \times F_{15,4}$	$F_{247,77}$	$S_{16} \times F_{63,8}$	$F_{255,16}$	S_{16}	$F_{288,17}$	$S_{18} \times F_{80,9}$	$F_{352,111}$	$S_{20} \times F_{24,5}$
17	$S_{11} \times F_2$	$F_{264,109}$	$F_{273,64}$	$F_{280,99}$	$F_{285,134}$	$F_{288,17}$	S_{17}	$F_{323,18}$	$F_{357,169}$	$F_{391,137}$
18	$S_{11} \times F_2$	$S_{18} \times F_{8,3}$	$F_{299,116}$	$S_68 \times F_{77,34}$	$S_{18} \times F_{35,6}$	$S_{18} \times F_{80,9}$	$F_{323,18}$	S_{18}	$F_{360,19}$	$S_{20} \times F_{99,10}$
19	$S_{11} \times F_2$	$S_{12} \times F_2$	$F_{325,51}$	$F_{336,71}$	$F_{345,91}$	$F_{352,111}$	$F_{357,169}$	$F_{360,19}$	S_{19}	$F_{399,20}$
20	$S_{11} \times F_2$	$S_{12} \times F_2$	$S_{13} \times F_2$	$S_54 \times F_{91,27}$	$S_{20} \times F_{15,4}$	$S_{20} \times F_{24,5}$	$F_{391,137}$	$S_{20} \times F_{99,10}$	$F_{399,20}$	S_{20}

Table 19: Colorings that realize the number of colors in Table 18.

Proof.

If. If m divides n, then $S_m \times F_{1,n}$ is a sufficient coloring with mn colors, and therefor it is a sufficient coloring, and so $C(m,n) = mn$.

Only if. Assume then $m < n$, and that we have a sufficient coloring using mn colors. Consider a $(m+1) \times (n+1)$ rectangle.

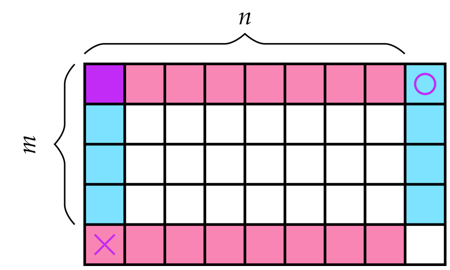


Figure 139: Proof

Let's say the color in the top-left corner is purple. All the colors in the leftmost n columns of the top row will be called "shades of red", and all the colors in the top m columns of the left column will be called "shades of blue", as shown in Figure 139. Purple is both a shade of red and a shade of blue.

We now look at row $m+1$. The only colors available for the leftmost n columns are shades of red. Furthermore, as $m < n$, the leftmost square in row $m+1$ cannot be purple, as this would cause a vertical rectangle with the same upper-left corner to have two purple squares. With only the shades of red available for that row, purple must appear somewhere else among the leftmost n columns in row $m+1$.

On the other hand, in column $n+1$ we can only use shades of blue, among which there must be a purple square. If the circled square does *not* use the color purple, then the lower-right $(m \times n)$ rectangle has two purple squares. Hence the circled square must be purple. Thus two squares at distance n along the same row must have the same color. Repeating the argument with rows and columns interchanged shows that two square at distance n along a column have the same color.

We have shown that if two squares lie in the same row or the same column and are exactly n squares apart on that row or column, then they must both have the same color. Note that since no intervening squares on that row or column can also have the same color, this means that every row and every column is basically colored periodically, with period n.

We next prove that m divides n.

Suppose that m does not divide n, but we have an mn-coloring. This mn-coloring is determined by its values on an $n \times n$ square. Let C_i be the set of colors used on the ith row of this square. We see that $C_1, \ldots, C_m$ are pairwise disjoint (these rows all being contained within an $m \times n$ rectangle), and that $C_i = C_{m+i}$ for all $i < n - m$, since C_{m+i} must be disjoint from $C_{i+1}, \ldots, C_{m+i-1}$, leaving only the n colors in C_i available. (Row $m+i$ and row i might have these colors in a different order, but they will be the same set of colors.)

If m divided n, then we'd get each of the sets $C_1, \ldots, C_m$ appearing exactly n/m times on the square. However, since m doesn't divide n, this repeating pattern of sets gets "cut off" at the bottom, and C_1 appears on some row C_{n-i} for $i < m$. Now a horizontal rectangle starting at row $n - i$ will contain two rows colored using colors from C_1 once the square repeats, contradicting the hypothesis that this is a sufficient coloring.

Therefor, if we have a sufficient coloring using mn colors, m must divide n. □

Theorem 148 (Puleo (2018b)[2]). *For $m > 1$, $C(m, m+1) = m(m+2)$.*

[2] Some details of the proof have been provided by Peter Taylor; see the reference for more information.

[Referenced on pages 135, 139 and 141]

Proof. Consider a $m \times (m+1)$ rectangle. Let all the top colors be *shades of red*, and the right-most color *crimson*.

Let all the colors *not* in the rectangle be shades of yellow. The cells below the rectangle must either be shades of yellow, or shades of red. But consider now a $m \times (m+1)$ rectangle with its top right corner just to the left of the crimson cell. We can see the cells must all be crimson or a shade of yellow, except the right-most one that must be purple. By moving this rectangle left, we can see the only one of those that can be crimson is the left most.

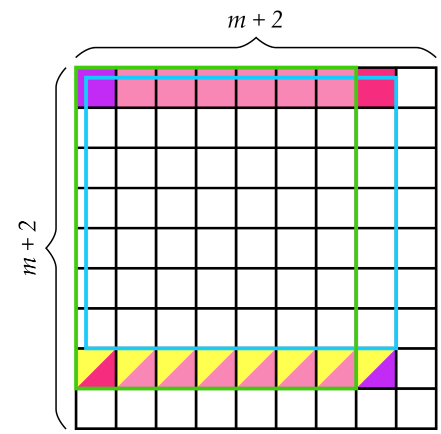


Figure 140: Proof

Now suppose we have less than m shades of yellow. This means neither the left-most or right-most cell cannot be yellow. They must be crimson and purple respectively. This means we have a periodical coloring, with a cell the same color as those with offset $(\pm m, \pm m)$.

Therefore the tiling is periodic: every mth row consists of the $m+1$ reds and $m1$ yellows shifted by m from the row m before. But none of the rows in between can contain any red or yellow, so we end up requiring $m(2m) = 2m^2$ colors. But if we have less than m shades of yellow, we have less than $m(m+1) + m$ colors (the colors inside the purple rectangle plus the yellows), i.e. we have less than $m^2 + 2m$ colors. But for $m > 1$, this is less than $2m^2$, which is a contradiction. Therefor we need at least m shades of yellow, and so we need at least $m(m+2)$ colors in total.

But we also have $C(m, m+1) \leq m(m+2)$ (Theorem 150), and since we need at least $m(m+2)$ colors, we need exactly $m(m+2)$ colors.

The coloring $F_{m(m+1)-1, m(m+2)}$ is sufficient, and since it uses $m(m+2)$ colors it is also optimal. □

Theorem 149. *Suppose P is a scaled copy of Q, and the scaling factor is a. Then $C(P) = n^2 C(Q)$. If K is the optimal coloring for Q, then $K \times S_a$ is the optimal coloring for P.*

[Not referenced]

Proof. □

Theorem 150. *Assume $m \leq n$. Then $C(m,n) \leq (m')^2 k$ where $m' \geq m$ and $m'k \geq n$, with $S_{m'} \times F_k$ a sufficient coloring.*

[Referenced on page 138]

Proof. If $m' \geq m$ and $m'k \geq n$, then $R(m,n) \subset R(m', m'k)$, and so $C(m,n) \leq C(m', m'k)$ (Theorem 140), and so $C(m,n) = (m')^2 k$ (Theorem 147). □

Theorem 151 (Taylor (2018))**.** $C(am, an) \leq a^2 C(m,n)$

[Not referenced]

Proof. Let K be an optimal coloring for $R(m,n)$. Define $K'(x,y) = a^2 K(\lfloor \frac{x}{a} \rfloor, \lfloor \frac{x}{a} \rfloor) + (x \bmod a) + a(y \bmod a)$. This coloring uses $a^2 C(m,n)$ colors.

It is easy to see K' is sufficient for $R(am, an)$, and therefor $C(am, an) \leq a^2 C(m,n)$. □

Theorem 152. $C(k, 2k-1) = 2k^2$

[Referenced on pages 135 and 139]

Proof. $R(k, 2k-1) \subset R(k, 2k)$, so $C(k, 2k-1) \leq C(k, 2k) = 2k^2$. So all we need to show is that $2k^2 - 1$ colors are not enough.

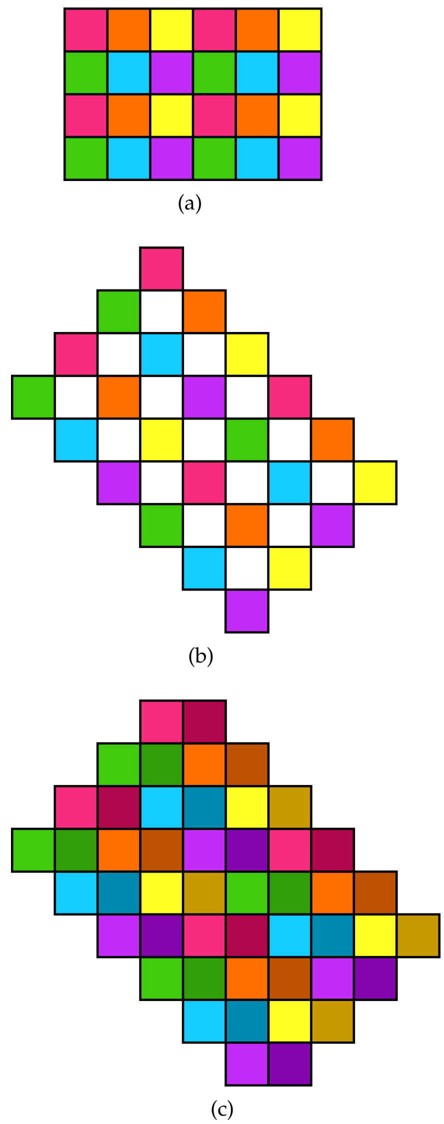
Towards a contradiction, suppose it *is* enough, and that K is an optimal coloring. Let $\delta_{xy} = x + y \bmod 2$, and define a new coloring as follows:

$$K'(x,y) = \begin{cases} K\left(\frac{x+y}{2}, \frac{x-y}{2}\right) & \text{for } (x+y) \equiv 0 \pmod 2 \\ K'(x-1, y) + k^2 - 1 & \text{for } (x+y) \equiv 1 \pmod 2 \end{cases}$$

Notice that

- K' has twice the number of colors as K, so it has $4k^2 - 2$ colors.
- Two cells (x,y) and (x', y') can only have the same color if $x + y \equiv x' + y' \pmod 2$.
- If two cells have the same color in K', then two corresponding cells have the same color in K.

We will now show K is an sufficient coloring for $R(2k, 2k-1)$. Suppose it is not; then there are two cells that can be covered with $R(2k, 2k-1)$. These cells must satisfy $x + y \equiv x' + y' \pmod 2$, and WLG assume $x + y \equiv x' + y' \equiv 0 \pmod 2$.



(a) (b) (c)

Figure 141: An example showing the coloring transformation used in Theorem 152. The first image show (an example) of original coloring K. The second image the new coloring for cells with $x + y$ (other cells are left blank). The final image shows the complete new coloring, with a darker shade of a color i used for color $i + k^2 - 1$.

Now if (x, y) and (x', y') lies inside the same $R(2k, 2k-1)$, then a and b lie inside the same $R(k, 2k-1)$, which means K cannot be sufficient, a contradiction. Therefor, K' must be sufficient.

Now K' uses $4k^2 - 2$ colors, but by Theorem 148 $C(2k, 2k-1) = 4k^2 - 1$; this is impossible (no sufficient coloring can use less colors than the chromatic number, which is minimal by definition). This means then that $2k^2 - 1$ colors is not enough for $R(k, 2k-1)$, which is what we wanted to show.

Therefor, $C(k, 2k-1) = 2k^2$. □

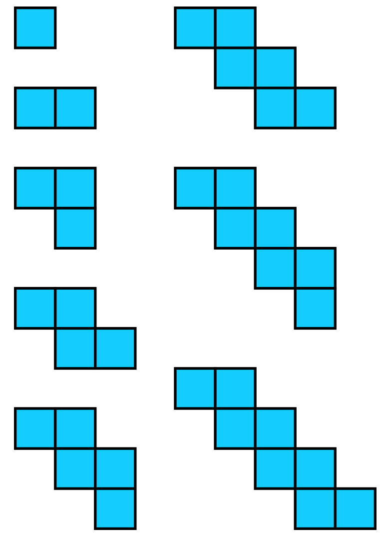


Figure 142: The first 8 W-polyominoes

Problem 42. *The theorem above "scaled" the coloring above by a factor of two, so that we could deduce constraints on the optimal colorings for $R(k, 2k-1)$ from $R(2k, 2k-1)$ (rectangles double the size).*

It is worth investigating why this idea cannot be modified to work in other cases:

- *We can the double coloring not work to make deductions for $R(2k, 2k+1)$?*
- *Why can we not use the trick above twice, for example, to deduce something for $R(k, 4k+1)$ from $R(2k, 4k+1)$?*
- *The transformation preserved squares (in the sense that a square of colors from the same colors will transform into a square of same colors in the new coloring). Why is this necessary?*
- *There are not square-preserving transformations that are not simply scale-transformations for all factors. What factors are possible?*
- *Why can other factors than 2 not work; for example, to try to deduce something for $R(k, 5k-1)$ from $R(5k, 5k-1)$?*
- *We can use a forward and backward translation to get rational factors that does not have the problem of other integer factors. Why can this idea not work?*

Problem 43 (Conjecture[3]). *Prove that for $m \leq n$*

$$C(m,n) = \begin{cases} mn - m^2 & \text{if } m < \sqrt{2}n \\ m^2 \left\lceil \frac{n}{m} \right\rceil & \text{otherwise.} \end{cases}$$

Note that we already proved it for when $m|n$, when $n = m-1$, and when $n = 2m-1$.

A general **W-polyomino**, which we will denote by W_n, is a snake with n cells that alternate in two directions. [4] W-polyominoes are weak. W polyominoes with odd area are unusual because they are not efficient.

[3] Since formulating this conjecture I discovered that it is not completely true. The first part seems true (when $m < \sqrt{2}n$, but the last part is false in between a third and a half of cases.

[4] Scherphuis (2016) discovered that the set of tiles $W_1, W_2, \cdots W_8$ make an interesting puzzle. The set tiles many symmetric figures, including $R(6,6)$, $R(4,9)$, $B(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8)$, $B(3 \cdot 4 \cdot 5 \cdot 6^2 \cdot 5 \cdot 4 \cdot 3)$, $B(5 \cdot 6 \cdot 7^2 \cdot 6 \cdot 5)$. He calls these polyominoes *zig-zag polyominoes*.

Theorem 153. *For W_n, and let $k = \lceil n/2 \rceil$. We need $2k$ colors, and an optimal coloring is given by $F_{2k,k}$.*

[Referenced on page 141]

Proof. It is clear that $F_{2k,k}$ is has enough colors. Since it has $2k$ colors, and $2k = n$ for n even, it follows that $C(P) = n$ for n even. It remains to show that for odd n, when $2k = n + 1$, we need $n + 1$ colors.

Since $W_n \subset W_{n+1}$, it follows that $C(W_n) \leq C(W_{n+1})$ (Theorem 140), so for odd n, $C(W_n) \leq C(W_{n+1}) = n + 1$. It is therefor enough to show that n colors are not enough.

Let's assume we need only n colors. Place the W-polyomino anywhere, and color the cells that it covers from 0 to $n - 1$.

Now place the W-polyomino so that it covers all the same cells except the one with color 0. The uncolored cell must be of color 0 (since we have only n colors and all the other colors are used already).

Repeat with color 1.

We can continue in this way, but for this proof we need to go only this far.

Now put the W-polyomino so that it covers the first few cells with even color, but none with odd colors. The cells that are not colored yet must be colored with the odd colors, one of each (again, because we only have n colors). In particular, one of the cells must be color 1. It cannot be the cell next to color 0, since it lies in the same W shape as the cell below which also has color 1. Therefor, one of the remaining cells must be color 1. Those cells are marked orange in Figure 146.

Finally, move the W-polyomino one cell down and one to the right. It now covers all three orange cells (one of which must be color 1), and another cell with color 1. This violates the condition of this being a legal coloring, which means, we cannot get away with only n colors.

□

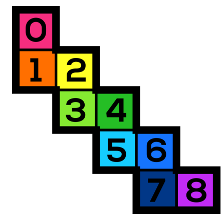


Figure 143: Coloring the first cells.

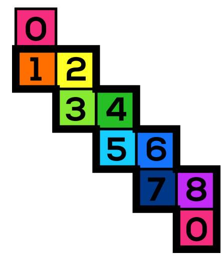


Figure 144: One color is forced.

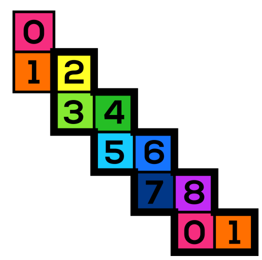


Figure 145: Another color is forced.

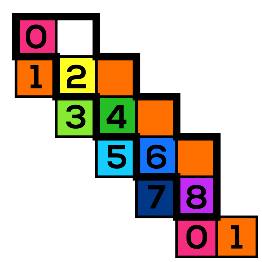


Figure 146: One of the unnumbered orange cells must be color 1.

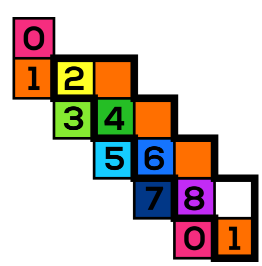


Figure 147: Two of the same colors (color 1) must occur in the same W. Therefor, we need at least one color more.

Problem 44. *Show that*

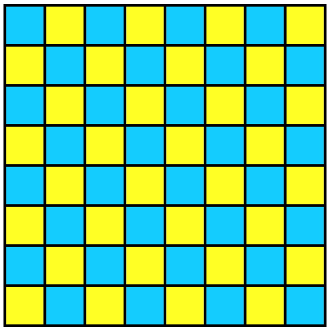
(1) *the only efficient pentominoes are the I- and X-pentominoes;*

(2) *the only efficient hexominoes are the I- and W-hexominoes; and*

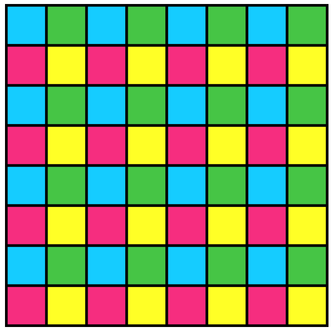
(3) *the only efficient heptominoes is the I-heptomino.*

While efficient polyominoes are relatively scarce for polyominoes with 7 or less cells, there are *lots* with 8 or 9 colors; the problem below explores why.

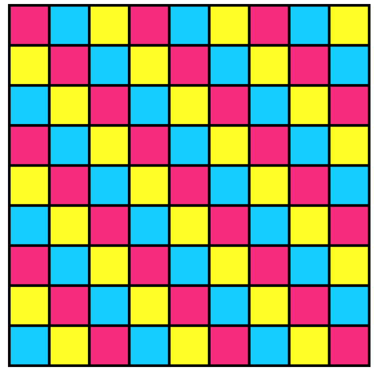
Problem 45.



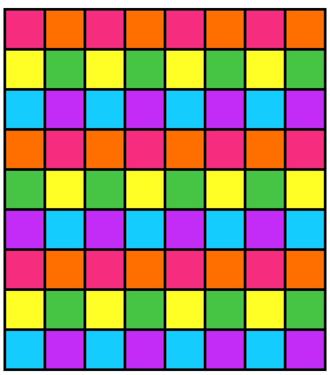
(a) Checkerboard coloring (F_2)



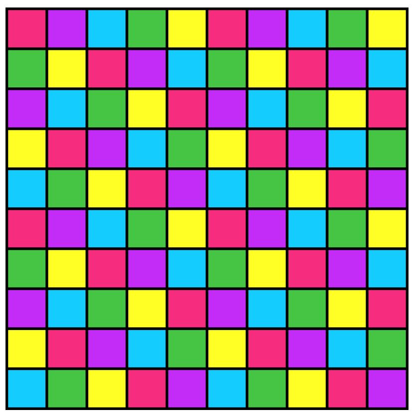
(b) S_2



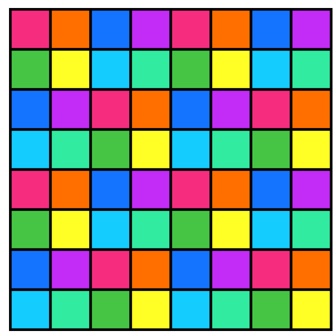
(c) F_3



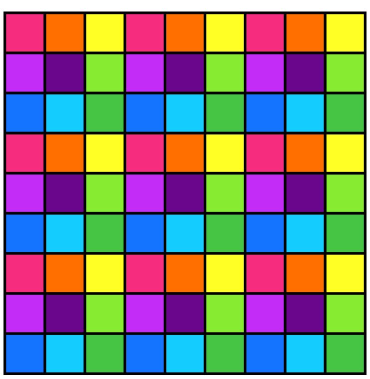
(d) $F_{6,3}$



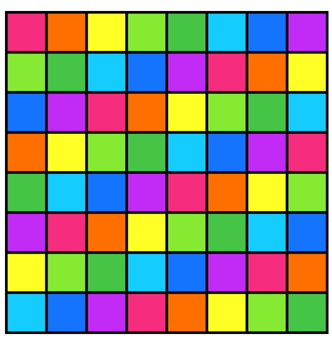
(e) $F_{5,2}$



(f) $S_2 \times F_2$



(g) S_3



(h) $F_{8,5}$

Figure 148: Some standard colorings.

(1) *Find a coloring $F_{n,k}$ that is not the optimal coloring of any efficient polyomino.*

(2) *Show that any polyomino formed from k^2 cells of different colors from a square coloring S_k has that square coloring as an optimal coloring, and hence such an polyomino is efficient.*

(3) *Show that any polyomino formed from $2k^2$ cells of different colors from a coloring $F_2 \times S_k$ has that coloring as an optimal coloring, and hence such an polyomino is efficient.*

(4) *Find examples that show the above in general does not hold for polyominoes with mk^2 cells of different colors from $F_{m,n} \times S_k$.*

(5) *How many octominoes are efficient?*

(6) *Can you say which decominoes are efficient?*

Problem 46.

(1) *Is every optimal coloring periodic?*

(2) *Does every polyomino have an optimal coloring of the form $F_{m,n} \times S_k$?*

(3) *We considered colorings where each polyomino cover each color at most once. What about colorings where each polyomino covers each color at most k times? At least k times?*

(4) *Complete the tables by finding the exact chromatic number for the unknown hexominoes and heptominoes.*

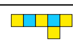
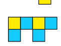
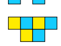
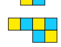
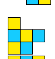
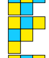
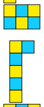
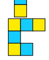
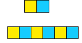
Polyomino P	$C(P)$		Theorems
Monomino	1	F_1	147(1)
Domino	2	F_2	147(1)
Bar	3	F_3	147(1)
Right	4	S_2	141
Bar	4	F_4	147(1)
Square	4	S_2	147(2)
T	5	$F_{5,2}$	139 ($Q = X_5$), 143 ($Q = X_5$)
L	8	$S_2 \times F_2$	141 ($Q = R(3,2)$), 137
Skew	4	$F_{4,2}$	153
I	5	F_5	147(1)
L	8	$S_2 \times F_2$	141
P	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
Y	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
N	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
W	6	$F_{6,3}$	153
F	8	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
V	9	S_3	141
T	8	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
U	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
X	5	$F_{5,2}$	144
Z	9	$S_2 \times F_2$	139 ($Q = R(3,3)$), 137
	[8, 12]		
	8	$S_2 \times F_2$	139 ($Q = L_4$), 137
	8	$S_2 \times F_2$	139 ($Q = L_4$), 137
	8	$S_2 \times F_2$	139 ($Q = L_4$), 137
	(8)	$S_2 \times F_2$	140 ($Q = T_5$)
	9	S_3	140 ($Q = V_5$), 137
	8	$S_2 \times F_2$	148
	12	$S_2 \times F_3$	141
	9	S_3	140 ($Q = V_5$), 137
	6	F_6	147(1)

Table 20: Chromatic numbers for small polyominoes. Intervals $[a, b]$ denote lower and upper bounds. Numbers in brackets show the upper bound is provided by an explicit coloring rather than a theorem.

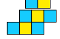
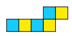
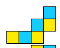
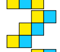
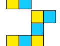
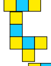
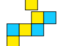
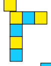
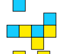
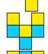
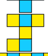
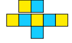
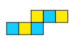
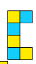
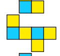
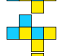
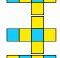
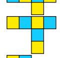
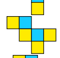
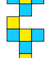
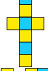
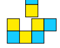
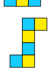
Polyomino P	$C(P)$		
	9	S_3	140 ($Q = Z_5$), 137
	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
	8	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	(9)	S_3	139 ($Q = V_5$), Coloring
	6	$F_{6,3}$	153
	(9)	S_3	140 ($Q = V_5$), Coloring
	(9)	S_3	140 ($Q = V_5$), Coloring
	(9)	S_3	140 ($Q = Z_5$), Coloring
	15	$F_{15,11}$	141
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	9	S_3	140 ($Q = Z_5$), 137
	(8)	$F_{8,5}$	140 ($Q = L_4$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
	[9, 15]		
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	[9, 10]	($F_{10,7}$)	
	(9)	S_3	139 ($Q = V_5$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	8	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring

Table 21: Chromatic numbers for small polyominoes.

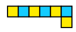
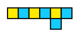
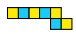
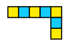
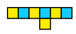
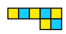
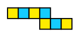
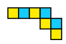
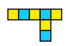
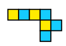
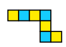
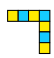
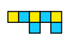
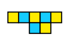
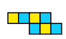
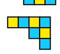
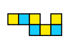
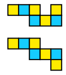
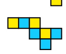
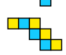
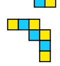
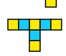
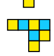
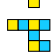
Polyomino P	$C(P)$		
	7	F_7	147(1)
	12	$S_2 \times F_3$	141
	12	$S_2 \times F_3$	140 ($Q = L_6$), 137
	12	$S_2 \times F_3$	140 ($Q = L_6$), 137
	18	$S_3 \times F_2$	141
	[8, 12]		
	12	$S_2 \times F_3$	140 ($Q = L_6$), 137
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	[8, 18]		
	[15, 18]		
	(15)	$F_{15,4}$	140 ($Q = L(3,4)$), Coloring
	(15)	$F_{15,4}$	140 ($Q = L(3,4)$), Coloring
	16	S_4	141
	12	$S_2 \times F_3$	140 ($Q = L(2,5)$), 137
	[8, 12]		
	(8)	$S_2 \times F_2$	140 ($Q = L(2,5)$), Coloring
	15	$F_{15,4}$	140 ($Q = L(3,4)$), 137
	[8, 12]		
	[8, 18]		
	[8, 18]		
	[8, 18]		
	[8, 16]		
	[9, 18]		
	[9, 15]		
	[9, 15]		

Table 22: Chromatic numbers for small polyominoes.

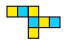
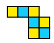
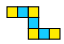
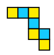
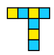
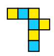
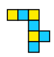
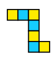
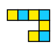
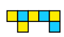
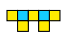
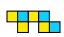
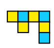
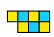
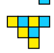
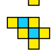
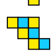
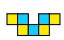
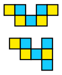
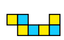
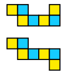
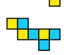
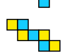
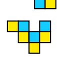
Polyomino	Class		
	(9)	S_3	140 ($Q = V_5$), Coloring
	(9)	S_3	140 ($Q = V_5$), Coloring
	(9)	S_3	140 ($Q = V_5$), Coloring
	[9, 16]		
	[15, 16]		
	(15)	$F_{15,4}$	140 ($Q = L(3,4)$), Coloring
	(15)	$F_{15,4}$	140 ($Q = L(3,4)$), Coloring
	(15)	$F_{15,4}$	140 ($Q = L(3,4)$), Coloring
	15	$F_{15,4}$	140 ($Q = L(3,4)$), 137
	12	$S_2 \times F_3$	140 ($Q = L(2,5)$), 137
	[8, 12]		
	(8)		140 ($Q = L_4$), Coloring
	15	$F_{15,4}$	140 ($Q = L(3,4)$), 137
	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
	(9)	S_3	140 ($Q = V_5$), Coloring
	[9, 15]		
	(9)	S_3	140 ($Q = V_5$), Coloring
	(9)	S_3	140 ($Q = V_5$), Coloring
	[8, 12]		
	[8, 15]		
	[8, 12]		
	[8, 18]		
	[8, 18]		
	(9)	S_3	140 ($Q = V_5$), Coloring
	[8, 15]		

Table 23: Chromatic numbers for small polyominoes.

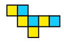
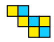
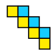
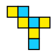
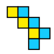
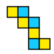
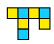
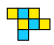
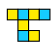
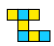
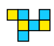
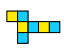
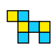
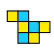
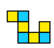
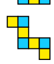
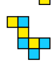
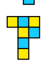
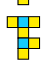
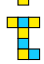
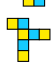
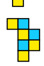
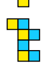
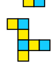
Polyomino P	$C(P)$		
	[8, 18]		
	(9)	S_3	140 ($Q = V_5$), Coloring
	8	$F_{8,4}$	153
	[8, 16]		
	[8, 16]		
	[9, 16]		
	[9, 15], 12?		
	[8, 15]		
	(9)	S_3	140 ($Q = V_5$), 137
	[9, 15]		
	(9)	S_3	140 ($Q = V_5$), Coloring
	[9, 15]		
	[8, 18]		
	(9)	S_3	140 ($Q = V_5$), Coloring
	(9)	S_3	140 ($Q = V_5$), Coloring
	(9)	S_3	140 ($Q = V_5$), Coloring
	[9, 16], 15?		
	(9)	S_3	140 ($Q = V_5$), Coloring
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	(9, 15)		
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	[9, 16]		
	[9, 15]		
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	[11?, 16]		

Table 24: Chromatic numbers for small polyominoes.

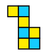
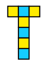
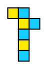
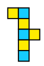
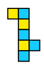
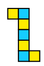
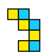
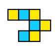
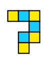
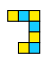
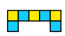
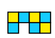
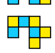
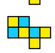
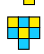
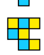
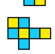
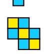
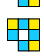
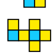
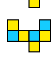
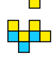
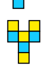
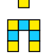
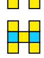
Polyomino	Class		
	[9, 15]		
	[12, 18]		
	[12, 18]		
	[12, 18]		
	[12, 18]		
	[12, 18]		
	[8, 15]		
	[8, 9]		
	[9, 15]		
	15	$F_{15,4}$	140 ($Q = L(3,4)$), 137
	12	$S_2 \times F_3$	140 ($Q = L(2,5)$), 137
	8	$S_2 \times F_2$	140 ($Q = L_4$), 137
	[9, 15]		
	[8, 15]		
	8	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	9	S_3	140 ($Q = Z_5$), 137
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	9	S_3	140 ($Q = V_5$), 137
	9	S_3	140 ($Q = V_5$), 137
	[8, 15]		
	[9, 15]		
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	8	$S_2 \times F_2$	
	9	S_3	140 ($Q = V_5$), 137
	9	S_3	140 ($Q = V_5$), 137

Table 25: Chromatic numbers for small polyominoes.

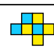
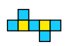
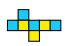
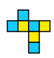
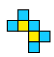
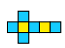
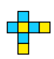
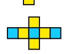
Polyomino P	$C(P)$		
	(8)	$S_2 \times F_2$	140 ($Q = L_4$), Coloring
	[9, 18]		
	[8, 18]		
	[9, 16]		
	[9, 16]		
	[8, 18]		
	[9, 16]		
	[8, 18]		

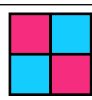
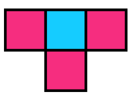
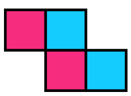
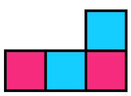
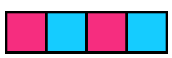
Table 26: Chromatic numbers for small polyominoes.

5.3 *Border Words*

Simply-connected polyominoes can be uniquely defined by their border; and the border can be described as a set of unit steps in one of four directions. Let's use x for horizontal steps, and y for vertical steps. And let's use an exponent for the amount and direction; positive for right and up, negative for left and down. Finally, let's decide to always go anticlockwise around the polyomino with the inside on the left. We call this sequence of steps a **word**.

With this, one way to describe a monomino is $xyx^{-1}y^{-1}$. We can also start with another vertex, giving us a cyclically shifted word such as $yx^{-1}y^{-1}x$. We will not distinguish between words cyclically shifted. Fig 27 shows the border words for the tetrominoes.

Table 27: Border words for tetrominoes.

Polyomino	Border word
	$x^2y^2x^{-2}y^{-2}$
	$xyxyx^{-3}y^{-1}xy^{-1}$
	$x^2yx{-}1yx^{-2}y^{-1}xy^{-1}$
	$x^3y^2x^{-1}y^{-1}x^{-2}y^{-1}$
	$x^4yx{-}4y^{-1}$

Problem 47.

(1) *What is the border word of $R(m,n)$?*

(2) *What is the border word for $B(m^a \cdot n^b)$?*

(3) *Prove sum of exponents of x equals the sum of exponents of y equals* 0.

(4) *Suppose we have a word to describe a segment $x^a y^b$; what is the word that describes the same segment traveled in reverse?*

The **reverse** of a word $W = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_n}y^{b_n}$ is defined as $\hat{W} = y^{-b_n}x^{-a_n}\cdots y^{-b_2}x^{-a_2}y^{-b_1}x^{-a_1}$; that is, we reverse the order of the steps and change the direction of all steps.

If W is the border word of a polyomino, then $\hat{W}$ is the word of an infinite region with a hole the same shape as the original polyomino.

Problem 48. *Suppose $W = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_n}y^{b_n}$ is a polyomino. What happens if we*

(1) *scale all exponents of x by a factor k?*

(2) *scale all exponents by a factor k?*

(3) *change the signs of all exponents of x?*

(4) *change the signs of all exponents of x and y?*

(5) *swap x and y?*

Can you write conditions on the border word of a polyomino for which type of symmetry the polyomino has?

6
Rectangles

We saw that determining whether a region can be tiled by dominoes can be determined efficiently. But in general,this is not the case. In fact, it is hard to determine if a polyomino will even tile a rectangle.

Consider the right tromino. It is easy to see it will tile $R(2,3)$, and thus all rectangles $R(2m,3n)$. Are these the only rectangles that it will tile? It turns out we can also tile $R(5,9)$, so and so also $R(5m,9n)$. Have we found all the rectangles it will tile? Not yet! For we can stack $R(2,6)$ and $R(3,6)$ together to tile $R(5,6)$, which adds $R(5m,6n)$ to the mix. Combined with the $R(5m,9n)$, this allows us to tile $R(5m,3n)$ for $n \geq 2$. And combining this in turn with $R(2m,3n)$ allows us to tile $R(m,3n)$ for $m = 2$ and all $m > 3$ and $n > 2$. Thus, we can tile almost all rectangles with area divisible by three (the only exceptions are $R(m,3)$ for odd m — which are in fact not tileable as we will see later).

The tromino case was easy enough to analyze. We found two rectangles, called *prime rectangles*, from which all other tilings can be derived. But in other cases, the analysis is much more difficult. For example, the Y-pentomino requires a set of 40 rectangles before we can construct the set of all rectangles.

And what is really surprising, is how difficult it is to know which rectangles can be tiled by a set of rectangles. This is the topic of the first section.

The minimum number of polyominoes that can tile a rectangle is called the *order* of the polyomino. We know very little about what orders are possible. We know orders 2 and 4 are possible (and in fact ubiquitous), and we know order 3 is impossible. But we do not know whether 5, or 6, or 7 are possible. We know all orders of the form $4k$ are possible, but other than that, the highest order polyomino we know is 246. This is the topic of the second section.

The third section deals with prime rectangles and proofs of polyominoes that don't tile rectangles.

The fourth section deals with fault-free tilings of polyominoes.

While we have a nice solution for when the polyomino is a rectangle, in general the picture is murky. The easy cases are usually those with a small number of prime rectangles.

The final section deals with simple tilings — tilings that have no subtilings with more than one tile that are rectangles. There are simple tilings of rectangles by any number of pieces larger than for, except for 6. (This is the second time 6 shows up out of the blue as an exception — recall that R(m, n) has fault free tilings form $m, n > 4$, except for $R(6, 6)$. This is probably a coincidence.)

6.1 *Tilings by Rectangles*

6.1.1 *Which rectangles can be tiled by a set of rectangles?*

Many problems can be broken into simpler problems. One natural way to approach a polyomino tiling problem is to break regions into rectangles. Rectangles are the simplest figures, and therefor we can say a lot about them. This section is primarily about how rectangles can tile other rectangles. (So, if we know that a polyomino tiles certain rectangles, we can answer which other rectangles it may tile.) Unfortunately, this is a rather hard problem to solve in general.

In this section throughout $\mathcal{T} = \{R(p_i, q_i)\}$ is a set of rectangles, and $R = R(m, n)$ is some rectangle we wish to tile using tiles from the set *without rotation*. If a tiling problem allows for rotation, simply add a rotated copy to the set of rectangles. We will prove a bunch of theorems; Table 28 at the end of the section summarizes which theorem applies in each situation.

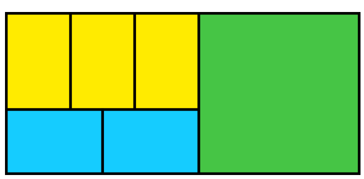


Figure 149: The smallest rectangle that requires all three tiles from the set $R(2, 3)$, $R(3, 2)$ and $R(5, 5)$.

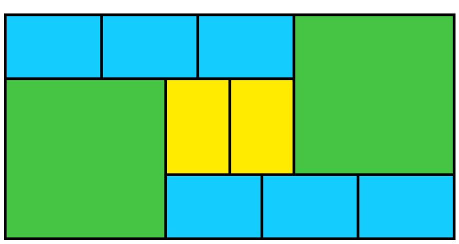


Figure 150: The smallest "interesting" rectangle that requires all three tiles from the set $R(2, 3)$, $R(3, 2)$ and $R(5, 5)$.

We begin by stating obvious necessary conditions. Let $\mathcal{T} = \{R(p_i, q_i)\}$. Then $\mathcal{T}$ can tile $R(m, n)$ when all the following conditions are met:

(1) $m = \sum p_i a_i$ for some integers $a_i \geq 0$ (Theorem 17)

(2) $n = \sum q_i b_i$ for some integers $b_i \geq 0$ (Theorem 17)

(3) $mn = \sum p_i q_i c_i$ for some integers $c_i \geq 0$ (Theorem 1)

Example 16.

- *If $\mathcal{T} = \{R(2,3), R(3,2)\}$, and $R = R(5,5)$, then conditions (1) and (2) hold, but not (3).*
- *If $\mathcal{T} = \{R(2,3), R(3,3)\}$, and $R = R(3,5)$, then conditions (1) and (3) hold, but not (2).*

The following theorem is trivial; I state it for completeness.

Theorem 154 (Bower and Michael (2004), Observation, Section 1)**.** *Let $\mathcal{T} = \{R(p,q)\}$. A tiling exists if and only if $p \mid m$ and $q \mid n$.*

[Referenced on pages 157, 158, 161, 201 and 251]

Proof.

If. Suppose $m = pk$ and $n = q\ell$. A tiling is given by placing ℓ rows with k tiles in each row.

Only if. All sides of the rectangle are mountains, so necessity follows directly from Theorem 17. □

Theorem 155 (Klarner (1969), Theorem 5)**.** *Let $\mathcal{T} = \{R(p,1), R(1,p)\}$. A tiling exists if and only if p divides m or n.*

[Referenced on pages 156, 157 and 205]

Proof.

If. Suppose WLG that p divides m, so that $m = pk$. We can put k bars in a row, and stack n of them to form a tiling of $R(m,n)$.

Only if. Apply the flag coloring with p colors (that is, each cell with coordinates x, y gets the color $(x+y)$ mod p).

No matter how we place the bars, each covers one each of the p colors. So if the tiling requires k tiles, a tiling will have k of each color.

Now suppose m mod $p = m' > 0$ and n mod $p = n' > 0$, and WLG that $m' \leq n'$. We can divide the rectangle into 4 quadrants: $R(m-m', n-n')$, $R(m-m', n')$, $R(m', n-n')$ and $R(m', n')$. The first three have tilings by the bars (by the first part of the theorem) and so each have the same number of each color. The last rectangle, however, has m' of color 0, but $m'-1$ of color 1 (see Figure 151). Therefor, a tiling is impossible, and therefor either m' or n' must be 0, and so either m or n must be divisible by p. □

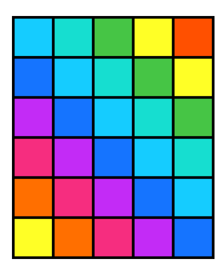


Figure 151: An example with $m' = 5$, $n' = 7$, and $p = 8$. There are 5 cells of light blue, but only 4 cells of green-blue.

Theorem 156 (Klarner (1969), Corollary of Theorem 5). *Let $\mathcal{T} = \{R(p,q), R(q,p)\}$. A tiling exists if and only if each of p and q divide either m or n, and if pq divides one of side, the other side can be expressed as $px + qy$ with $x, y \geq 0$.*

[Referenced on pages 164 and 165]

Proof.

If. To prove there is a tiling when the conditions hold, there are two cases to consider:

(1) $p|m$ and $q|n$

(2) $p|m$, $q|m$ and $n = px + qy$.

Suppose $p < q$. The first case is trivial: simply place m/p rectangles in n/q rows. For the second case: a tiling is given by placing x rows of m/p rectangles vertically, and y rows of m/q rectangles horizontally.

Only if. If $R(m,n)$ has a tiling by $\mathcal{T}$, then it must also have a tiling by $R(p,1)$ and $R(1,p)$, and so by Theorem 155 we must have p divides m or n.

Similarly, it must also have a tiling by $R(q,1)$ and $R(1,q)$, and so q divides m or n.

Each side is expressible in the form $px + qy$ by either Theorem 17 or 18.

□

Theorem 157 (Slightly generalized from Martin (1991), p.43). *$R(m,n)$ is tileable by $R(p,1)$ and $R(1,q)$ iff p divides m or q divides n.*

[Referenced on page 157]

Proof. *If.* WLG suppose p divides m. A tiling is given by stacking $R(p,1)$ in each row.

Only if. WLG, assume p does not divide m. We will show q divides n.

Color the columns of $R(m,n)$ cyclically with p colors. Denote the number cells of color i by c_i. Since p does not divide m, we can write $m = dp + r$, where $0 < r < p$. There is $d+1$ columns of the first color, and d columns of the second color; therefor $c_0 = n(d+1)$, and $c_{p-1} = nd$.

Consider now a tiling by the two bars. The horizontal bar covers one of each color; the vertical bar covers q cells of the same color. Let x be the number of horizontal bars, and y_i the number of vertical bars that cover cells of color i.

Then $c_0 = x + qy + 0$, and $c_{p-1} = x + qy_{p-1}$, so $c_0 - c_{p-1} = q(y_0 - y_{p-1})$.

Now

$$\begin{aligned} n &= n(d+1) - hd \\ &= c_0 - c_{p-1} \\ &= q(y_0 - y_{p-1}), \end{aligned}$$

which means q divides n, which is what we wanted to show. □

Theorem 158 (Divisibility Lemma, Bower and Michael (2004), Section 3)**.** *Suppose we have a set of rectangles partitioned into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that all rectangles in $\mathcal{T}_1$ have widths with a factor r and all rectangles in $\mathcal{T}_2$ have heights with a factor s. Then if $\mathcal{T}$ tiles a $R(m, n)$, then $r \mid m$ or $s \mid n$.*

[Referenced on pages 157, 158 and 161]

Proof. [1] The tiles in $\mathcal{T}_1$ are all tileable by $R(r, 1)$, and the tiles in $\mathcal{T}_2$ are all tileable by $R(1, s)$. Therefor, if a tiling of $R(m, n)$ by $\mathcal{T}$ exists, it also has a tiling by $\mathcal{T}' = \{R(r, 1), R(1, s)\}$.

But by Theorem 157, this is possible only if $r \mid m$ or $s \mid n$. □

[1] The original proof uses the idea of *charges*. The argument here hinges on Theorem 157, where we adapted an argument Martin originally used to prove Theorem 155 (and is in fact very similar to the argument we used there from Golomb). Clearly these three theorems are closely related.

Theorem 159 (de Bruijn (1969), Theorem 1)**.** *Suppose a number k divides one side of each rectangle in $\mathcal{T}$. If $\mathcal{T}$ tiles $R(m, n)$, then either $k \mid m$ or $k \mid n$.*

[Referenced on pages 160, 161, 165 and 223]

Proof. This follows immediately from 158 by setting $r = s = k$. □

One consequence of this theorem is that we cannot tile a 10×10 with horizontal and vertical bars of length 4.

Theorem 160 (Bower and Michael (2004), Theorem 5)**.** *Let p_1, p_2, q_1, and q_2 be positive integers with $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$. Then $R(m, n)$ can be tiled by translates of $R_1 = R(p_1, q_1)$ and $R_2 = R(p_2, q_2)$ rectangular bricks if and only if one of the following holds:*

(1) *$p_1 \mid m$ and $q_1 \mid n$*

(2) *$p_2 \mid m$ and $q_2 \mid n$*

(3) *$p_1 p_2 \mid m$ and $n = aq_1 + bq_2$ for some integers a, b*

(4) *$q_1 q_2 \mid n$ and $m = ap_1 + bp_2$ for some integers a, b.*

[Referenced on pages 161 and 185]

Proof. [2]

If. In case (1), a tiling of $R(m,n)$ by $R(p_1,q_1)$ exists by Theorem 154; similarly in case (2), a tiling of$R(m,n)$ by $R(p_2,q_2)$ exists.

In case (3), suppose $m = m'p_1p_2$. We can split the rectangle in $R(m'p_1p_2, aq_1)$ and $R(m'p_1p_2, bq_2)$. From Theorem 154, the first is tileable by R_1 and the second by R_2.

In case (4), suppose $n = n'q_1q_2$. We split the rectangle in $R(ap_1, nq_1q_2)$ and $R(bp_2, nq_1q_2)$, which by Theorem 154 is tileable by R_1 and R_2 respectively.

Only if. If R is tileable by $R(p_1,q_1)$ and $R(p_2,q_2)$, then $m = ap_1 + bp_2$ for some integers a, b and $n = a'q_1 + b'q_2$ for some integers a', b' (Theorem 17 or 18). Since R_1 has width p_1, and R_2 has height q_2, by the divisibility lemma (Theorem 158) we have p_1 divides m or q_2 divides n. Similarly, R_2 has width p_2 and R_1 has height q_1, so p_2 divides m or q_1 divides n. Taken together, one of the four conditions must hold. □

[2] Kolountzakis (2004) gives an alternative proof using the Fourier transform. Fricke (1995) showed the special case when the two rectangles are squares.

Problem 49. *Which rectangles can be tiled by $R(2,2)$ and $R(3,3)$?*

Example 17. *Let $\mathcal{T} = \{R(6,6), R(10,10), R(15,15)\}$. There are 6 different ways to partition this set so that the conditions for Theorem 158 hold. These give the following set of conditions, which must be satisfied for any tileable rectangle $R(m,n)$:*

(1) $2 \mid m$ *or* $15 \mid n$

(2) $3 \mid m$ *or* $10 \mid n$

(3) $5 \mid m$ *or* $6 \mid n$

(4) $15 \mid m$ *or* $2 \mid n$

(5) $10 \mid m$ *or* $3 \mid n$

(6) $6 \mid m$ *or* $5 \mid n$

From this we can make inferences such as the following:

- *If $m = 6$, it already satisfies 1, 2, and 6. Since none of 5, 15, or 10 divides 6, the only way to satisfy conditions 4, 5 and 6 is for $6 \mid n$.*
- *For a tileable rectangle $R(m,n)$, we have one of 6, 10 or 15 must divide either m or n.*
- *If m is prime, then $30 \mid n$.*

It is not hard to see the conditions are sufficient for a tiling to exist provided the sides satisfy Theorem 17.

Example 18. *Let $\mathcal{T} = \{R(3,2), R(2,3), R(9,5), R(5,9)\}$. These rectangles have a common factor 3, so we know they will only tile rectangles with a factor of 3.*

We can stack three of the smaller rectangles to make $R(9,2)$ and $R(2,9)$. By combining these with $R(9,5)$, we $R(5,9)$, we can tile $R(9,k)$ or $R(k,9)$ for any $k \neq 1,3$.

We can also tile $R(6,2), R(2,6)$, and we can also tile $R(6,3)$ and $R(3,6)$, so we can tile any $R(6,k)$ and $R(k,6)$ for any $k > 1$. (Because any $k >$ can be written as $2x + 3y$ for nonnegative integers x and y.)

Finally, we can tile all $R(3,2k)$ and $R(2k,3)$ for $k \geq 1$. Putting this together, we have the following:

(1) *$R(3m,n)$ for all $m > 1$, $n > 3$*

(2) *$R(3,2n)$ for all $n \geq 1$*

(3) *$R(2m,3)$ for all $m \geq 1$*

Theorem 161 (Fricke (1995)). *Let a,b, x and y be positive integers with $\gcd(x,y) = 1$. Then, $R(a,b)$ can be tiled with $R(x,x)$ and $R(y,y)$ if and only if either*

(1) *a and b are both multiple of x, or*

(2) *a and b are both multiple of y, or*

(3) *one of the numbers a, b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.*

[Not referenced]

Things get more interesting when $\mathcal{T}$ has more than two rectangles. For rectangles satisfying certain conditions, they can tile any sufficiently large rectangle.

This is somewhat similar to what happens to integers. Given two integers that are relatively prime[3], we can write any big-enough integer as a positive combination of the two. For example, given integers 3 and 5, we can write any integer larger than 7 as the sum of multiples of 3 and 5. For example, $8 = 3 + 5$, $13 = 3 + 2 \cdot 5$, $11 = 2 \cdot 2 + 5$.

Of course, we can do the same when we have a larger set of integers. For example, given 3, 4 and 5, we can write any number larger than 3 as a positive combination. We call the minimum number that cannot be written as a positive combination of the set the set's **Frobenius number**, and we write it $g(a, b, c, \cdots)$. For example, $g(3,5) = 7$, and $g(3,4,5) = 2$.

Theorem 162 (Labrousse and Ramírez Alfonsín (2010), Theorem 5). *Suppose we have $k \geq 3$ rectangles R_i, where is R_i is a $p_i \times q_i$ rectangle, and*

[3] Recall, two integers are realtively prime when they don't have any common divisors other than 1

(1) $\gcd(p_i, p_j) = 1$, *for any* $i \neq j$,

(2) $\gcd(q_i, q_j) = 1$, *for any* $i \neq j$,

Let g_1 *be the maximum Frobenius number of* p_i *taken* $(k-1)$ *at a time. Let* g_2 *be the maximum Frobenius number of products* $\frac{q_1 \cdots q_k}{q_i q_j}$ *for some* j *and* $i \neq j$.

Then if we have $p_i, q_i \geq \max(g_1, g_2)$, *the rectangle is tileable by* R_i.

[Referenced on page 161]

The proof is quite technical, and I omit it. For details, see the reference. There is also a courser version of the theorom above, that is much easier to calculate:

Theorem 163 (Labrousse and Ramírez Alfonsín (2010), Corollary 1)**.** *Suppose we have* $k \geq 3$ *rectangles* R_i, *where is* R_i *is a* $p_i \times q_i$ *rectangle, and*

(1) $\gcd(p_i, p_j) = 1$, *for any* $i \neq j$,

(2) $\gcd(q_i, q_j) = 1$, *for any* $i \neq j$,

If $r = \max_i(p_i, q_i)$, *then the rectangle is tileable by this set if* $m, n > r^4$.

[Not referenced]

Rectangles	Minimum Length
$2 \times 3, 3 \times 2, 5 \times 5$	30

Theorem 164. *For a set of three rectangles, one of the following holds:*

(1) $\gcd(q_i, p_j) = \gcd(q_i, q_j) = 1$

(2) *We can partition the rectangles as in Theorem 159 (possibly, with the two factors equal).*

[Not referenced]

Theorem 165. *Let* $R_1, \cdots, R_4$ *be four rectangles that satisfy the following conditions:*

(1) $\gcd(p_1, p_2) = r > 1$

(2) $\gcd(p_3, p_4) = s > 1$

(3) $\gcd(q_i, q_j) = 1$, *for* $i, j = 1, 2, 3, 4$, $i \neq j$

(4) $\gcd(r, s) = 1$

The there exist C *such that these rectangles can tile any* $R(m, n)$ *if* $m, n > C$.

[Referenced on page 161]

Proof. Let $u = \text{lcm}(p_1, p_2)$ and $v = \text{lcm}(p_3, p_4)$. We can then build these rectangles:

(1) $u \times q_1$

(2) $u \times q_2$

(3) $v \times q_3$

(4) $v \times q_4$

Using the first two of these, we can build any $u \times x$ rectangle for large enough x (say $x > C_1$) and using the last two rectangles, we can build any $v \times y$ rectangle for large enough y (say $y > C_2$). Since $\gcd(u, v) = 1$, we can, when $x = y$, build any $z \times x$ rectangle for large enough z (say $z > C_3$, and of course we already have $x, y > \max(C_1, C_2)$. So we can tile any $z \times x$ rectangle with $z, x > \max(C_1, C_2, C_3)$.

And of course,

- $C_1 = g(q_1, q_2) = q_1q_2 - q_1 - q_2$
- $C_2 = g(q_3, q_4) = q_3q_4 - q_3 - q_4$
- $C_3 = g(u, v) = uv - u - v = \frac{p_1p_2p_3p_4 - p_1p_2s - p_3p_4r}{rs}$

and so

$$C = \max\left\{q_1q_2 - q_1 - q_2, q_3q_4 - q_3 - q_4, \frac{p_1p_2p_3p_4 - p_1p_2s - p_3p_4r}{rs}\right\}$$

□

Theorem 166. *For a set of four or more rectangles, one of the following holds:*

(1) $\gcd(p_i, p_j) = \gcd(q_i, q_j) = 1$ *for* $i \neq j$.

(2) *We can partition the rectangles as in Theorem 159 (possibly, with the two factors equal).*

(3) *We can select 4 rectangles that can tile a sufficiently large rectangle.*

[Not referenced]

It can be tricky to keep track of the theorems and know which one applies in a given situation. Table 28 summarizes to conditions and the theorems that apply.

We conclude this section with two special cases where the rectangles in the tileset are all squares.

$\lvert\mathcal{T}\rvert$	Conditions	Theorem
1		154
2	$\gcd(p_1, p_2) = 1$ $\gcd(q_1, q_2) = 1$	160
2	$\gcd(p_1, p_2) = k > 1$	159
≥ 3	$\gcd(p_i, p_j) = 1, i \neq j$ $\gcd(q_i, q_j) = 1, i \neq j$	162
3	$\gcd(p_1, p_2) = k > 1$ $\gcd(p_i, p_3) = 1, i \neq 3$	158
4	$\gcd(p_1, p_2) = r > 1$ $\gcd(p_3, p_4) = s > 1$ $\gcd(q_i, q_j) = 1, i \neq j$ $\gcd(r, s) = 1$	165
≥ 4	$\gcd(p_1, p_2 \cdots, p_k) = r$ $\gcd(q_k, p_{k+1} \cdots p_{\lvert\mathcal{T}\rvert}) = s$	158
≥ 4	$\gcd(p_i, p_j) = 1, i \neq j, i, j = 1,2,3,4$ $\gcd(q_i, q_j) = 1, i \neq j, i, j = 1,2,3,4$	162

Table 28: Characterization of tilings of rectangles by rectangles.

Theorem 167 (Labrousse and Ramírez Alfonsín (2010), Theorem 6). *All sufficiently large rectangles can be tiled with three squares whose side-lengths are pairwise relatively prime.* $R(a, a)$ *can be tiled with* $R(a_1, a_1)$, $R(a_2, a_2)$, *and* $R(a_3, a_3)$ *when*

$$a > \left(2 - \sum_1^3 \frac{1}{a_i}\right) \prod_1^3 a_i$$

or

$$a > 2a_1a_2a_3 - a_1a_2 - a_1a_3 - a_2a_3$$

[Not referenced]

For example, for squares with length 2, 3, and 5, this theorem gives us $a > 2.2.3.5 - 2.3 - 3.5 - 2.5 = 60 - 6 - 15 - 10 = 29$.

Theorem 168 (Labrousse and Ramírez Alfonsín (2010), Theorem 7). *Let* $p > 4$ *be relatively prime to 2, and 3. Then* $R(a, a)$ *can be tiled with* $R(2, 2)$, $R(3, 3)$ *and* $R(p, p)$ *if* $a \geq 3p + 2$.

[Not referenced]

6.1.2 The gap number of rectangular tilings

Recall that the *gap number* of a region R is the minimum number of monominoes of a tiling by $\mathcal{T}^+$, which is the tileset $\mathcal{T}$ plus a monomino. One would think that determining the gap number for a set of rectangles and a rectangular region is easy; however, it is not the case. We will give some partial results in this section.

Theorem 169. *Suppose $R(m, n)$ with $m, n < p$ and $m + n < p$ is a rectangle with the flag coloring F_p applied. Then one color does not occur at all.*

[Referenced on page 163]

Proof. The first row goes from color 0 to color $m - 1$; the second row from color $p - 1$ to color $m - 2$; and so on, until the last row, which goes from $p - n + 1$ to $m - n$.

In row i, we have the first $n - i$ colors and the last i colors; since i ranges from 0 to $n - 1$, all colors satisfy $0 \leq c \leq m - 1$, or $0 \leq c < m$ and $p - (n - 1) + 1 \leq c \leq p - 1$, or $p - n \leq c \leq p - 1$. From the last inequality and $m + n < p$, we get $m < c \leq p - 1$. So all color used are smaller than m or bigger than m; not are equal to m. So color m is not used. □

Theorem 170. *Suppose $R(m, n)$ with $m \bmod p + n \bmod p < p$. Then $G_p(m, n) \geq (m - m \bmod p)(n - n \bmod p)$.*

[Not referenced]

Proof. Partition the rectangle in four rectangles R_1, R_2, R_3, R_4 such that $R_4 = R(m \bmod p, n \bmod p)$. There is a color that does not occur in R_4. Each color occurs the same number of times in each of R_1, R_2, R_3; each color occurs $(m - m')(n - n')/p$ in R_1, $m'(n - n')/p$ in R_2, and $n'(n - n')/p$ in R_3. One color is missing in R_4 (Theorem 169), so that color occurs this many times in total:

$$
\begin{aligned}
&(m - m')(n - n')/p + m'(n - n')/p + n'(m - m')/p \\
=&(mn - m'n - mn' + m'n' + m'n - m'n' + mn' - m'n')/p \qquad (6.1) \\
=&(mn + m'n')/p.
\end{aligned}
$$

So we can fit at most that many bars, which means $G(m, n) \geq mn - p(mn - m'n')/p$, or $G(m, n) \geq m'n'$. □

Theorem 171 (Barnes (1979), Lemma 1). *Let $\mathcal{T} = \{R(1, p), R(p, 1)\}$. Then*

$$G_{\mathcal{T}}(R(m,n)) = \begin{cases} (m \bmod p)(n \bmod p) & \textit{if } m < p \textit{ or } n < p \textit{ or } m \bmod p + n \bmod p < p \\ (m - m \bmod p)(n - n \bmod p) & \textit{otherwise} \end{cases}$$

[Referenced on page 223]

Theorem 172. *Exactly* $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ *squares* $R(p,p)$ *can fit in* $R(m,n)$.

[Referenced on pages 164 and 194]

Proof. (Adapted from JimmyK4542 (2017).) Apply the square coloring S_p to $R(m,n)$. No matter how we place a $p \times p$ square, it always covers p of each color. Let c be the color at coordinates $(p-1, p-1)$. This color occurs $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ times, so we can fit at most this number of squares. □

Theorem 173. *Let* $\mathcal{T} = \{R(p,p)\}$. *Then*

$$G_{\mathcal{T}}(R(m,n)) = mn - (m - m \bmod p)(n - n \bmod p).$$

[Referenced on page 22]

Proof. By Theorem 172 we can fit at most $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ squares, which means the gap number satisfies $G \geq G_{\mathcal{T}}(R(m,n)) = mn - (m - m \bmod p)(n - n \bmod p)$. The naive tiling has a gap of this size, and so is optimal, and therefor the gap number is given by $G_{\mathcal{T}}(R(m,n)) = mn - (m - m \bmod p)(n - n \bmod p)$. □

Theorem 174. $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor / q$ *rectangles* $R(p,pq)$, *and no more, can fit in* $R(m,n)$.

[Referenced on page 164]

Proof. By Theorem 172 exactly $k = \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ squares $R(p,p)$ can fit in $R(m,n)$, this means at most $\frac{k}{q}$ of $R(p,pq)$ can fit in $R(m,n)$.

All we need to show is a tiling that uses $\frac{k}{q}$ tiles.

Consider the rectangle $R(p \lfloor m/p \rfloor, p \lfloor n/p \rfloor)$ that fits in $R(m,n)$. For this rectangle, a tiling exists by Theorem 156. This tiling uses $\frac{k}{q}$ tiles. Thus we can fit $\frac{k}{q}$ of $R(p,pq)$ in $R(m,n)$. □

Theorem 175. *Let* $\{R(p,pq), R(pq,p)\}$. *Then*

$$G_{\mathcal{T}}(R(m,n)) = mn - \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor p^2.$$

[Not referenced]

Proof. From Theorem 174 we can fit $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor / q$ rectangles, and since each has an area of $p^2 q$, the total covered area is at most $\left\lfloor \frac{n}{p} \right\rfloor p^2$ and therefor the gap is at least $G_{\mathcal{T}}(R(m,n)) = mn - \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor p^2$. □

6.1.3 *Tiling by Rectangles of other Regions*

In this section, we extend some of the results of the previous section to tiling of arbitrary simple figures by rectangles.

Theorem 176 (Csizmadia et al. (2004), Corollary 2.4)**.** *If a simply-connected region is tileable by bars of length k, the region has at least one side divisible by k.*

[Referenced on page 165]

Theorem 177 (Csizmadia et al. (2004), Corollary 2.3)**.** *If a simply-connected region is tileable by $m \times n$ and $n \times m$ rectangles, then it must have at least one edge whose length is divisible by m, and at least one edge whose length is divisible by n.*

[Not referenced]

Proof. Note that the region is tileable by both bars of length m, and by bars of length n. Therefor, by Theorem 176 one side is divisible by m, and one side is divisible by n. □

For example, a region is not tileable by dominoes if all the edges have odd length.

These theorems are generalizations of Theorem 156 and Theorem 159.

Problem 50. *Is the following true or false? If a simply-connected figure is tileable by a set of rectangles $\mathcal{T}$ that can be partitioned into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that all the rectangles in $\mathcal{T}_1$ has widths with factor r and all rectangles in $\mathcal{T}_2$ has heights with factor s, then the figure has either an edge with factor r, or an edge with factor s.*

6.2 *Order*

A polyomino that can tile a rectangle is called **rectifiable**.

The **rectangular order** of a polyomino is the smallest number k such that k copies of the polyomino tiles a rectangle. The **odd rectangular order** of a polyomino is the smallest odd number k such that k copies of the polyomino can tile the rectangle, if such a tiling exists. Polyominoes that has an odd order are called **odd**, otherwise they are called **even** (Klarner, 1965).[4] The **square order** is the minimum

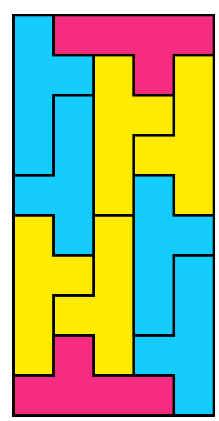


Figure 152: Polyomino with order 10.

[4] This notion of even is different from the one in Section 4.3.4.

number of tiles required to tile a square.

6.2.1 Order Theorems

Theorem 178. *If a polyomino P has order n, then a scaled copy of the polyomino P' order has order n.*

[Not referenced]

Proof. Suppose P' has order $n' < n$, and that the associated rectangle is R'. If we divide each tile into $S = k \times k$ squares, we have a tiling of R' by S. But this tiling is unique (Theorem 3), and is the trivial tiling with all squares lying in a grid. But if this is the case, we can scale the tiling by k, and so find a valid polyomino tiling of a rectangle using n' tiles, which means the order of P is smaller than n, a contradiction.

It follows that the order of P' must be larger than or equal to n. But by scaling a tiling of R by P, we can find a tiling of R' by P'. Therefor, the order of P is n. □

Theorem 179 (Klarner (1969), Theorem 2)**.** *An unbalanced polyomino with $2n$ cells is even.*

[Referenced on pages 190 and 196]

Proof. Since the polyomino has $2n$ cells, the rectangle must have an even area (Theorem 1), and is therefor balanced (see Theorem 52).

But an odd number of unbalanced polyominoes with an even number cannot have an equal number of white and black cells, so a region tiled by these cannot be balanced. Therefor, the only possibility for a balanced region be tileable by these polyominoes must be tileable by an even number of them. □

Theorem 180 (Klarner (1969), Theorem 4)**.** *Suppose the columns of a polyominoes are alternately colored black and white. Suppose the polyomino has k black cells or k white cells regardless of its position and orientation. Then the polyomino is even.*

[Not referenced]

Proof. Suppose R is a rectangle tiled by the polyomino. Since the area of the polyomino is even, so must the are of the rectangle (Theorem 1). WLG assume then the orientation of the rectangle is such that the rectangle has the same number of black and white cells; that is, the width is even.

Then form two sets, B with the copies of the polyomino colored with k black cells, and W with the copies of the rectangle with k

white cells. Clearly, $|B| = |W|$, and therefore the total number of polyominoes must be even. □

The **hull** of a polyomino is the smallest rectangle that can fit it. The hull of a rectangle is the rectangle itself.

Theorem 181. *A rectifiable polyomino must cover at least one corner of its hull.*

[Referenced on pages 167, 193 and 195]

Proof. There is no way to place the polyomino to cover a corner cell of the rectangle. □

This theorem is also a consequence Theorem 130, which states that any *reptile* must cover at least one corner of its hull.

Theorem 182 (Klarner (1965), (i) p. 18)**.** *If a polyomino has symmetry index 2 or less, and is rectifiable, it is a rectangle.*

[Not referenced]

Proof. The polyomino must cover at least one corner of its hull (Theorem 181).

Suppose the polyomino is in class **All**, **Rot2** or **Axis2**. Then, by symmetry, must cover all corners of its hull. If we place the polyomino in a corner of a rectangle, we cannot have gaps between the rectangle edge and the polyomino edge, therefor, all the cells between two adjacent corners must be part of the polyomino. This must be true for all edges of the polyomino, so we have a rectangle, possibly with some holes. But if there is a hole, it is too small to fit even one polyomino, so we cannot have holes. Therefor, the polyomino is a rectangle.

Suppose the polyomino is in class **Diag2**. If it covers two adjacent corners of its hull, by symmetry it covers 4 corners of the hull and the same argument as above applies.

If it does not cover two adjacent corners (Figure 153), by symmetry it must cover two opposite corners. If this polyomino tiles a rectangle, the tiles in the bottom corners must have different orientations (there is only one way to place a tile in each corner). If we consider the bottom edge, then there must be at least a pair of adjacent tiles with opposite orientations also adjacent to the rectangle edge. The hulls of these tiles cannot overlap, and so the cells corresponding to the hull corners that are not covered cannot be covered by another tile (See Figure 154). Therefor, this arrangement is impossible, and therefor this case cannot occur.

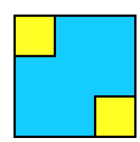


Figure 153: A polyomino (in blue) in **Diag2** that covers only 2 corners of its hull.

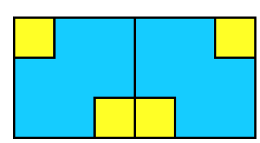


Figure 154: A polyomino (in blue) in **Diag2** that covers only 2 corners of its hull.

In all cases, the only possibilities are for polyominoes that are rectangles. □

Theorem 183 (Klarner (1965), (ii) p. 18)**.** *If a polyomino fits inside a rectangle, and covers two diagonally opposite corners of its this rectangle, and it tiles this rectangle, it is a rectangle.*

[Not referenced]

Proof. We have two cases: either the polyomino demarcates two disconnected sections, or it also covers a third corner of the hull. In both cases, we cannot fit the polyomino in the remaining regions to be tiled; so therefor, the only possibility is that the polyomino is a rectangle. □

Theorem 184 (Reid (2014), p. 117)**.** *If a polyomino tiles a (different) odd polyomino with an odd number of tiles, it is odd.*

[Not referenced]

Problem 51. *Does the converse hold?*

Theorem 185. *An order 2 polyomino must cover at least 2 adjacent corners of its hull.*

[Referenced on page 170]

Proof. Any dissection of a rectangle into two connected pieces must have at least one tile with two adjacent corners of the rectangle. Because if this is not the case, then pairs of opposite corners must belong to the same tile (Figure 6.2.1). But there is no way that the two blue corners can be connected *and* the two pink corners can be connected. Furthermore, all the corners of the rectangle of a piece must lie on that piece's hull.

□

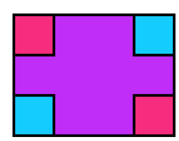


Figure 155: No dissection of a rectangle into two pieces can make pairs of opposite corners belong to the same pieces.

Figure 6.2.1 shows it is possible for a order-2 polyomino to only cover 2 corners of its hull.

Theorem 186. *A polyomino that covers exactly two opposite corners of its hull, with the bottom one on the left, can only tile a rectangle if it and a copy rotated* 90° *counter-clockwise can tile a straight edge.*

[Referenced on pages 190, 193 and 195]

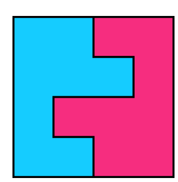


Figure 156: An example of a order-2 polyomino that covers only two of its hull's corners.

Proof. There are up to eight possible orientations for a polyomino. Let's call a polyomino with a covered corner on the bottom left *left*, and *right* otherwise. Suppose the polyomino tiles a rectangle, and consider the polyominoes making the bottom edge of the rectangles. Since the corners of this edge must be filled, we must have a left polyomino on the left and a right polyomino on the right. And so, on this edge, we must have at least one pair of left and right polyominoes adjacent.

The right polyomino cannot be the left polyomino reflected about the Y-axis. If this was the case, the top covered corners would be adjacent, and so we could not covered the uncovered corners, and so there would be a gap on the edge.

By similar reasoning, the right polyomino cannot be the left polyomino reflected about the Y-axis and then rotated 180°.

Next, suppose the right polyomino is the left polyomino rotated 90° clockwise.

If they are to pack an edge, their convex hulls must overlap, because if they don't, there is no way to tile the uncovered hull corners.

The center of rotation must lie within this overlapping hull.

Now consider the two corners of the overlapping rectangle along the edge. The left one must belong to L, (otherwise it would cover R's hull), and therefor the right cell must belong to R. If we rotate the left cell by 90° counter clockwise about the center of rotation, it overlaps with the right cell, which cannot be by Theorem 20. Therefor, this configuration is impossible.

That leaves only one possibility, namely, the right polyomino is rotated 90° counter-clockwise. Figure 6.2.1 shows that in this configuration it is possible for the two polyominoes to pack an edge, and indeed, the polyomino can tile a rectangle.

□

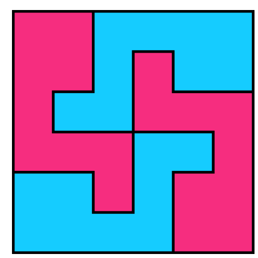


Figure 157: The only way in which a polyomino that covers only 2 opposite corners of it's hull can pack an edge with a copy. If such a polyomino can tile an edge, it has order 4.

Theorem 187. *If a n-omino and a* 90° *rotated copy pack an edge of length* $k \leq \sqrt{n}$*, the polyomino has order at most 4. And the order is exactly four when the polyomino covers exactly two opposite corners of the hull.*

[Not referenced]

Proof. By Theorem 20 we can add two more copies, rotated 180° and 270°degrees around the same center of rotation, and they won't overlap. In this compound figure, the packed edge must also exist in all four rotations; and together these form the border of a tiled rectangle (possibly with holes). So the four copies pack a rectangle (possibly with holes). Since the edge of length k, the rectangle must have area $k^2 \leq n$. So there cannot be any holes, and in fact $k^2 = n$.

Therefor, four copies tile a rectangle (with no holes), and so the order of the polyomino is at most 4.

If the polyomino covers exactly two corners of its hull, it cannot be a rectangle, so its order cannot be 1. It can also not have order 2 (since a order 2 polyominoes must at least cover two adjacent corners of its hull by Theorem 185). □

A **gun** is a bar graph with height 2, which we will draw upside down so that it resembles a gun (Figure 158).[5] A gun has a left and right **end**. If the corners are not missing from an end, it is called a **blunt end**. A gun with one blunt end is called a **blunt gun** (Figure 159); a gun with two blunt ends is a **degenerate gun** (Figure 160). The longest non-blunt end is called the **barrel**. If the other end is also non-blunt, it is called the **tab**. The side with the contiguous row of cells is called the **back** of the gun; the opposite side is called the **face**. A contiguous band of cells of the face is called a tooth, and the space between two teeth is called a **gap**. The number of cells in the tooth at the blunt end of a gun is the **thickness** of the blunt end. A gun that is a rectangle with two corners removed is called a **solid gun** (Figure 161).

[5] The terminology is from Dahlke, http://www.eklhad.net/polyomino/gun.html. He does not give this exact definition.

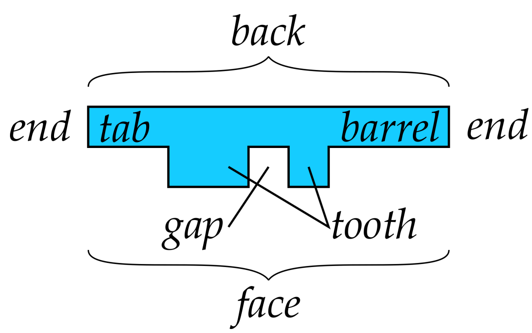


Figure 158: A gun is a bar polyomimo with height 2.

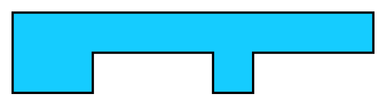


Figure 159: A blunt gun.

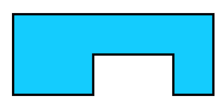


Figure 160: A degenerate gun.

Figure 161: A solid gun.

Theorem 188 (Dahlke)**.** *If a polyomino with height 2 is rectifiable, it is a gun.*[6]

[6] This is (1) from the *Gun Theorem*, http://www.eklhad.net/polyomino/gun.html.

[Referenced on pages 171 and 195]

Proof. A polyomino of width 2 must cover at least two corners of its hull.

If there are exactly two corners covered, and they are opposite, then there are only two ways to put the polyomino in a corner. Either way, we have gaps along the two rectangle edges that can only be filled in one of up to two ways; an all cases new gaps form. No matter how we continue along the edge, there are gaps, and so we can never fill an adjacent corner. Therefor, the polyominoes cannot tile a rectangle, so this case is not possible.

If these are the only two, or all four corners are covered, then the cells between them must be covered, as can be seen by placing the tile in the corner of the rectangle. We have a bar of height 2.

Suppose there are three corners covered, with the bottom right corner missing, and some squares from the top row. The first tile must cover the origin in exactly this orientation. The barrel points to the right. If it has more than two cells, it forces a second copy, rotated 180°, to fit in the space under the barrel. Since there are squares missing from the top, this isolates a hole that cannot be tiled (Figure 162).

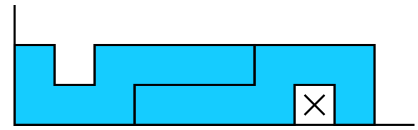


Figure 162:

If the barrel has length 1, and the gun is missing the second cell from the top row, the second gun can fill the hole under the barrel with its blunt end.

The gun at $(0,2)$ must be vertical, otherwise it creates holes with the first tile. If the barrel is up, and the third cell in the top row of the gun is covered, this causes an untileable hole (Figure 163). If the third cell is not missing, we can only cover $(1,1)$ by a vertical gun reflected and moved down and right. But both copies occupy $(1,n)$(Figure 164). Whatever the case, the barrel must point down and fills the gap at (1, 1).

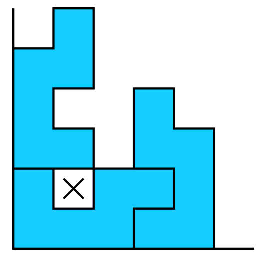


Figure 163:

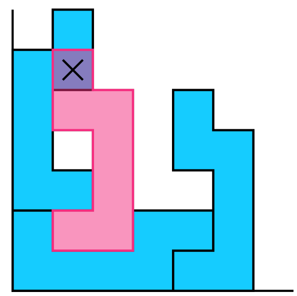


Figure 164:

But then either $(2,1)$ or $(2,2)$ cannot be covered without isolating untilable holes (Figures 165 and 166).

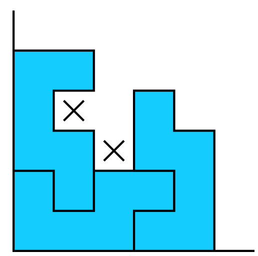


Figure 165:

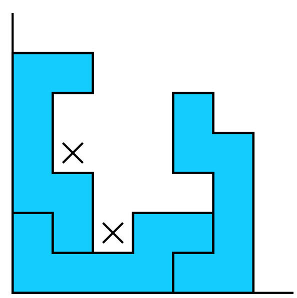


Figure 166:

Therefor, if three corners are covered, there cannot be any cells missing from the top row, and so we have a bar. □

Theorem 189 (The Gun Theorem, Dahlke)**.** *The only rectifiable polyominoes with width 2 and order higher than 2 are the T-tetromino, the Y-pentomino, the Y-hexomino, the D-hexomino, and the heptomino* $B(1 \cdot 2^2 \cdot 1^2)$*, except perhaps* $B(1 \cdot 2^3 \cdot 1^3)$*.*

[Referenced on page 195]

Proof. The proof is long. For full details, see the reference. Here is an outline:

In this proof, with "gun" we mean a gun that does not have order 2. Let n be the length of the gun.

(1) Theorem 188.

(2) If two guns are adjacent with their ends on the floor, they must be back-to-back.

(3) Three guns cannot be adjacent if they are all vertical. Therfore, we cannot fill 3 adjacent cells with guns with tabs. We can at most fill 4 cells with blunt guns.

(4) Gaps with width $n > k \geq 3$ and height 2 cannot be filled with guns with a tab, and gaps with width $n > k \geq 5$ and height 2 cannot be filled with blunt guns.

(5) Degenerate guns with two thick ends are not rectifiable.

(6) No thick ends can cover the origin.

(7) Degenerate guns with one thick end are not rectifiable.

(8) Degenerate guns with no thick ends are not rectifiable.

(9) Two guns aimed at the origin must cover $(1, 1)$.

(10) Guns with tabs of length larger than one are not rectifiable.

(11) In a tiling of a rectangle by blunt guns, the barrel cannot touch the blunt end,

(12) the blunt ends cannot touch, and

(13) the blunt edge cannot touch the barrel.

(14) Blunt guns that have barrels with more than one cell are not rectifiable.

(15) Blunt guns are not rectifiable.

(16) Solid guns are not rectifiable, except for the T-tetromino and the D-hexomino.

(17) A gun with a barrel of length 1 and teeth of lenth more than one are not rectifiable.

(18) A gun with a barrel of length 1 is not rectifiable.

(19) Guns with gaps are not rectifiable.

(20) Guns with a face of one cell are not rectifiable, except for the T-tetromino, Y-pentomino and Y-hexomino.

(21) Guns with a barrel with two cells are not rectifiable, except for the D-hexomino.

(22) Guns with a barrel with more than two cells are not rectifiable, except perhaps for $B(1 \cdot 2^3 \cdot 1^3)$ shown in Figure 167.

(23) This covers all guns, except for $B(1 \cdot 2^3 \cdot 1^3)$. It is not known whether this gun is rectifiable.

□

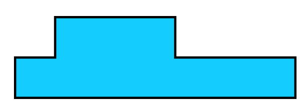


Figure 167: $B(1 \cdot 2^3 \cdot 1^3)$, the only polyomino with width 2 that we do not know whether it tiles a rectangle.

Problem 52 (Open Problem)**.** *Determine whether $B(1 \cdot 2^3 \cdot 1^3)$ (Figure 167) tiles a rectangle.*

6.2.2 *L-shaped polyominoes*

After Reid (2014), we make the following definitions: Let $L(a,b,c)$ be the polyomino which is made from two rectangles, $R(a,c)$ and $R(b,c)$, aligned along an edge as shown in Figure 168.

Two copies of these tiles the rectangle $R(a+b,2c)$, called the **basic rectangle**. We will assume that $\gcd(a,b,c) = 1$. This does not lead to any loss of generality since a $L(ad,bd,cd)$ is similar to $L(a,b,c)$.

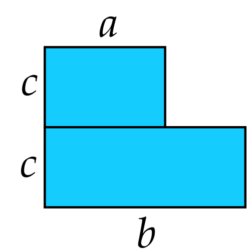


Figure 168: The polyomino $L(a,b,c)$ is made from $R(a,c)$ and $R(b,c)$.

Theorem 190 (Klarner (1969)). *$L(a,2a,b)$ is odd with odd order at most 15.*

[Not referenced]

Proof. Scale the tiling of $R(5,9)$ by the right tromino by (a,b) to find a new tiling of $R(5a,9b)$ or $R(9a,5b)$ by $L(a,2a,b)$. □

This gives us an infinite family of polyominoes with odd rectangular order at most 15 (Golomb, 1966, p. 104). Figure 169 shows an example where the tiling has been scaled by a factor of 3 horizontally and 2 vertically. It is not necessarily true that these polyominoes have odd order 15. For example, $L(1,2,1)$ has odd order 11 (Figure 180).

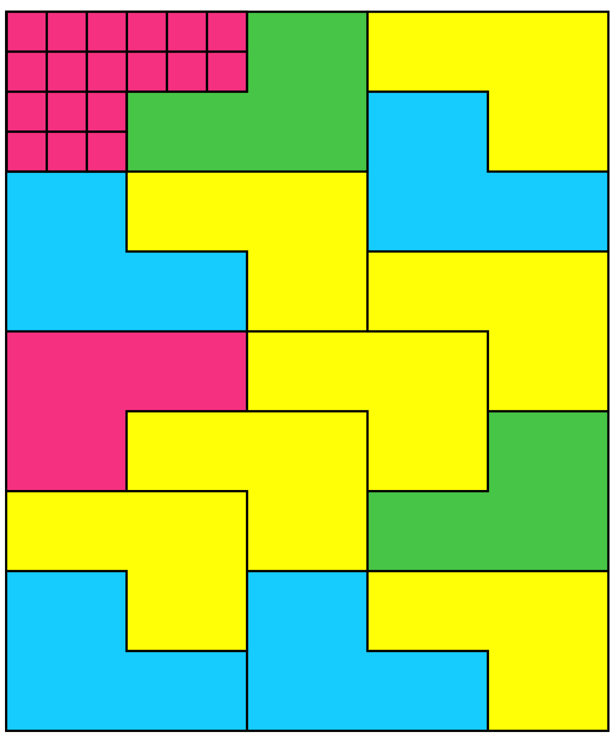


Figure 169: A infinite family of polyominoes with odd order at most 15 can be found by stretching the right tromino.

Theorem 191 (Fletcher (1996) via Reid (2014)). *$L(2s,3s,t)$ is odd if s is odd.*

[Not referenced]

Theorem 192 (Fletcher (1996) via Reid (2014))**.** *$L(2s, 3s, 1)$ is odd if s is even.*

[Not referenced]

Theorem 193 (Fletcher (1996) via Reid (2014))**.** *$L(s, 4s, t)$ is odd if s is odd.*

[Not referenced]

Theorem 194 (Fletcher (1996) via Reid (2014))**.** *$L(s, 4s, 1)$ is odd for all s.*

[Not referenced]

Theorem 195 (Fletcher (1996) via Reid (2014))**.** *$L(s, 4s, 3)$ is odd for all s.*

[Not referenced]

Theorem 196 (Jepsen et al. (2003) via Reid (2014))**.** *For odd n, $L(1, n-1, 1)$ tiles $R(n+2, 3n)$, and thus is odd with odd order at most $3(n+2)$.*

[Not referenced]

Theorem 197 (Jepsen et al. (2003) via Reid (2014))**.** *For $n \equiv 2 \pmod 4$, $L(1, n-1, 1)$ is odd.*

[Not referenced]

Theorem 198 (Reid (2014))**.** *Suppose a is odd, $2c$ divides b, and $\gcd(a, b, c) = 1$. Then $L(a, b, c)$ tiles $R((a+b)(a+2b)+b, 5c(a+b))$, and so is odd with order at most $5((a+b)(a+2b)+b)$.*

[Not referenced]

Theorem 199 (Reid (2014))**.** *If $\gcd(a+b, cd) = 1$, then $L(a, b, c)$ is odd.*

[Not referenced]

Theorem 200 (Reid (2014))**.** *If $L(1, k-1, 1)$ tiles $R(m, n)$, then either m is even, or $m \geq k-1$, and the same for n.*

[Not referenced]

Polyomino	OO	Rect
$L(1,5,1)$	21[ab]	9×14[c]
$L(3,4,1)$	33[ab]	11×21[c]
$L(4,5,1)$	49[ab]	21×21[c]
$L(2,3,2)$	33[ab]	15×22[c]
$L(1,4,2)$	45[ab]	18×25[c]
$L(1,5,2)$	35[ab]	20×21[a]
$L(2,5,1)$	57[a]	9×21[c]
$L(2,5,2)$	63[a]	18×49[a]
$L(2,3,3)$	55[a]	11×75[a]
$L(2,7,3)$	133[d]	57×63[d]
$L(4,5,3)$	119[d]	51×63[d]
$L(1,8,3)$	125[d]	45×45[d]

Table 29: Odd orders for various L-polyominoes. [a]Marshall (1997) [b]Reid (1997) [c]Reid [d]Reid (2014)

6.2.3 *Known Orders*

There are polyominoes of order $4k$ for any integer k. Table 30 summarizes results for other orders. There are no known polyominoes with rectangular order 6, but the tilings in Figures 170 and 209 suggest that such polyominoes can exist.

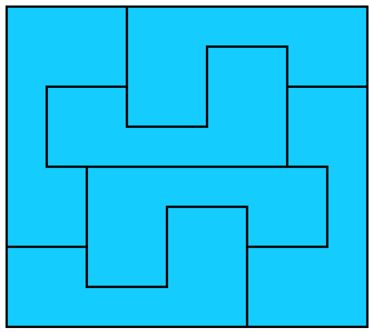


Figure 170: A rectangular tiling using six 12-ominoes. The polyomino has order 2.

Order	Tiling
1	All rectangles
2	Figure 179
4	Figure 185
10[a]	Figure 152
18[a]	Figure 172(a)
24[a]	Figure 172(b)
28[a]	Figure 173
50[a]	Figure 174
76[a]	Figure 176
60[b]	Figure 175
92[a]	Figure 177
96[a]	Figure 178
180[c]	http://www.cflmath.com/Polyomino/8omino10_rect.html
270[c]	http://www.cflmath.com/Polyomino/11omino7_rect.html
246[c]	http://www.cflmath.com/Polyomino/8omino11_rect.html

Table 30: Examples of orders. From [a]Golomb (1996, p. 97–100), [b]Grekov, http://polyominoes.org/rectifiable and [c]Reid, http://www.cflmath.com/Polyomino/rectifiable_data.html.

Golomb (1996) gives a tiling of an octomino with 312 copies, but this polyomino was later found to have order 246 (Reid, http://www.cflmath.com/Polyomino/8omino11_rect.html).

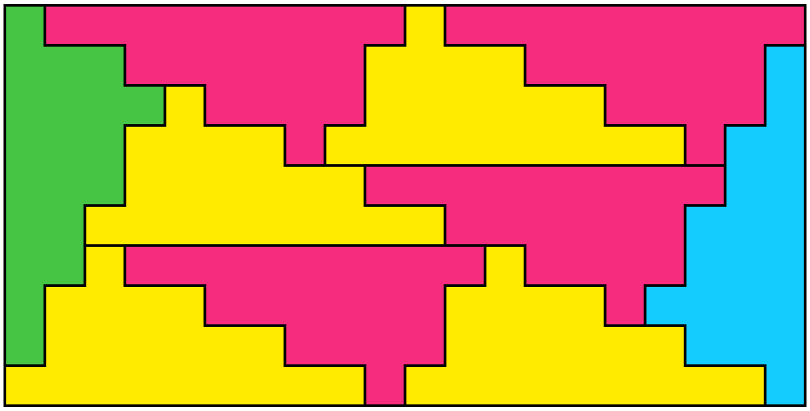
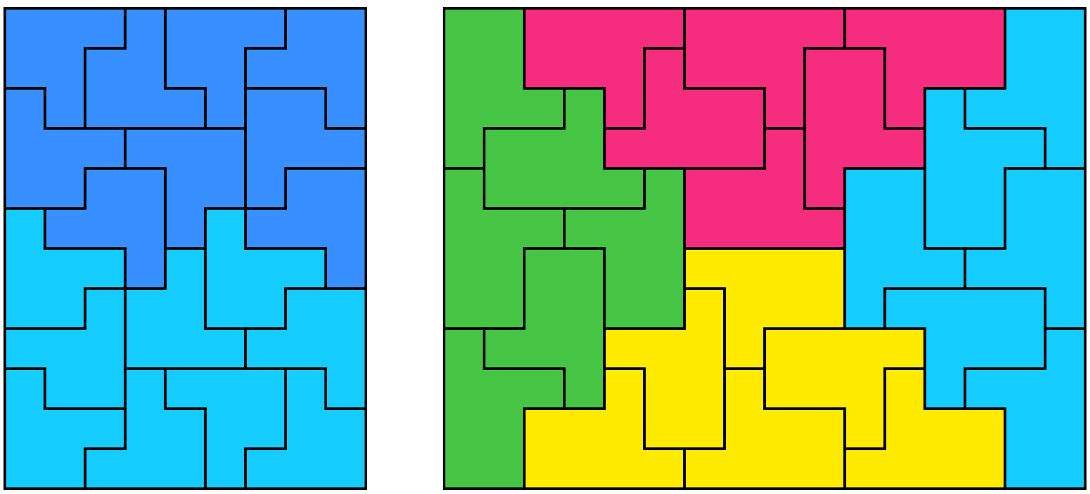


Figure 171: Marshall (1997).



(a) Polyomino of order 18.

(b) Polyomino of order 24

Figure 172: Golomb (1996).

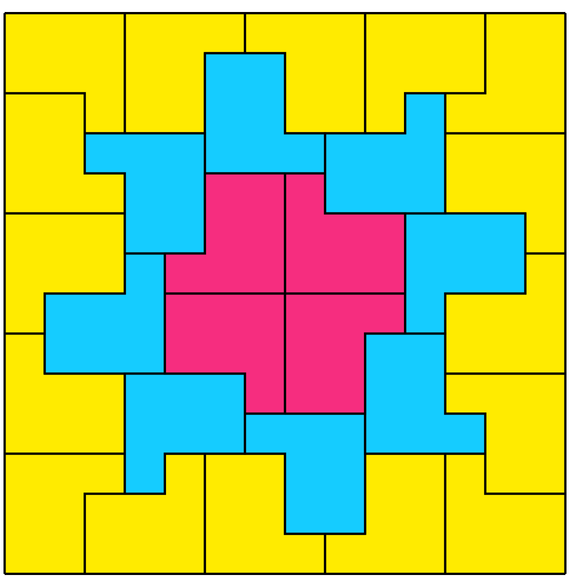


Figure 173: Polyomino with order 28.

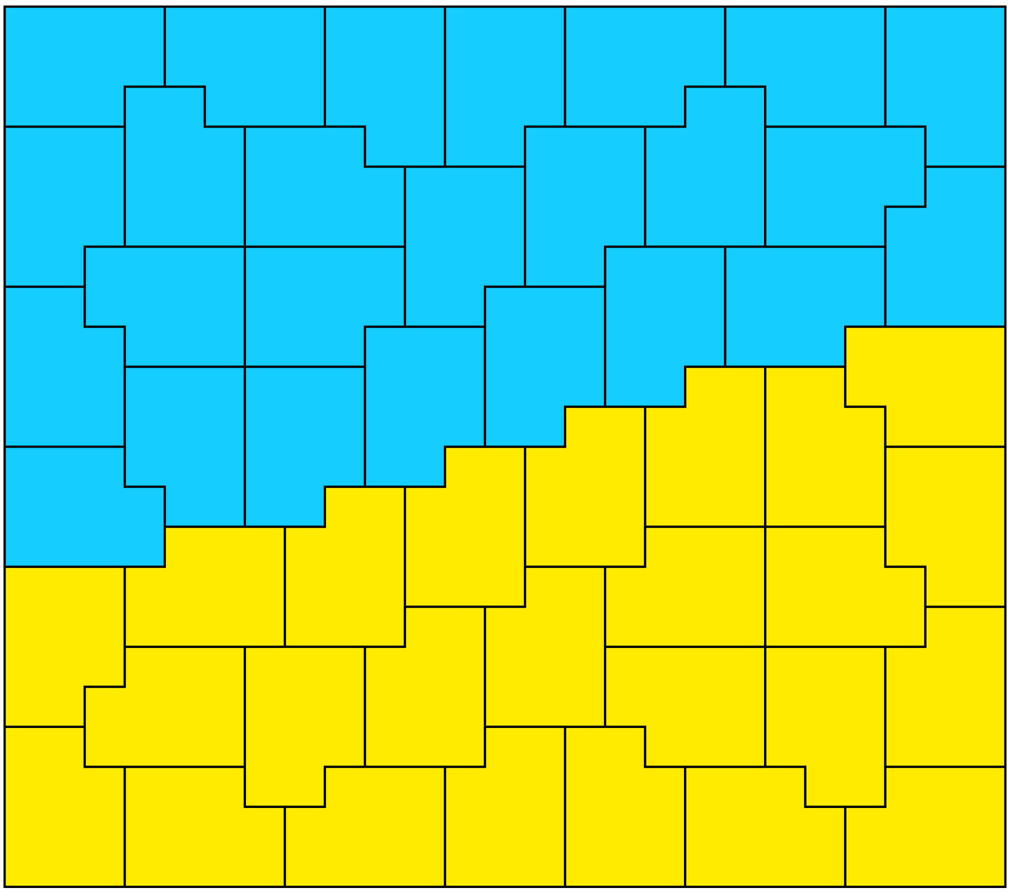


Figure 174: Polyomino with order 50.

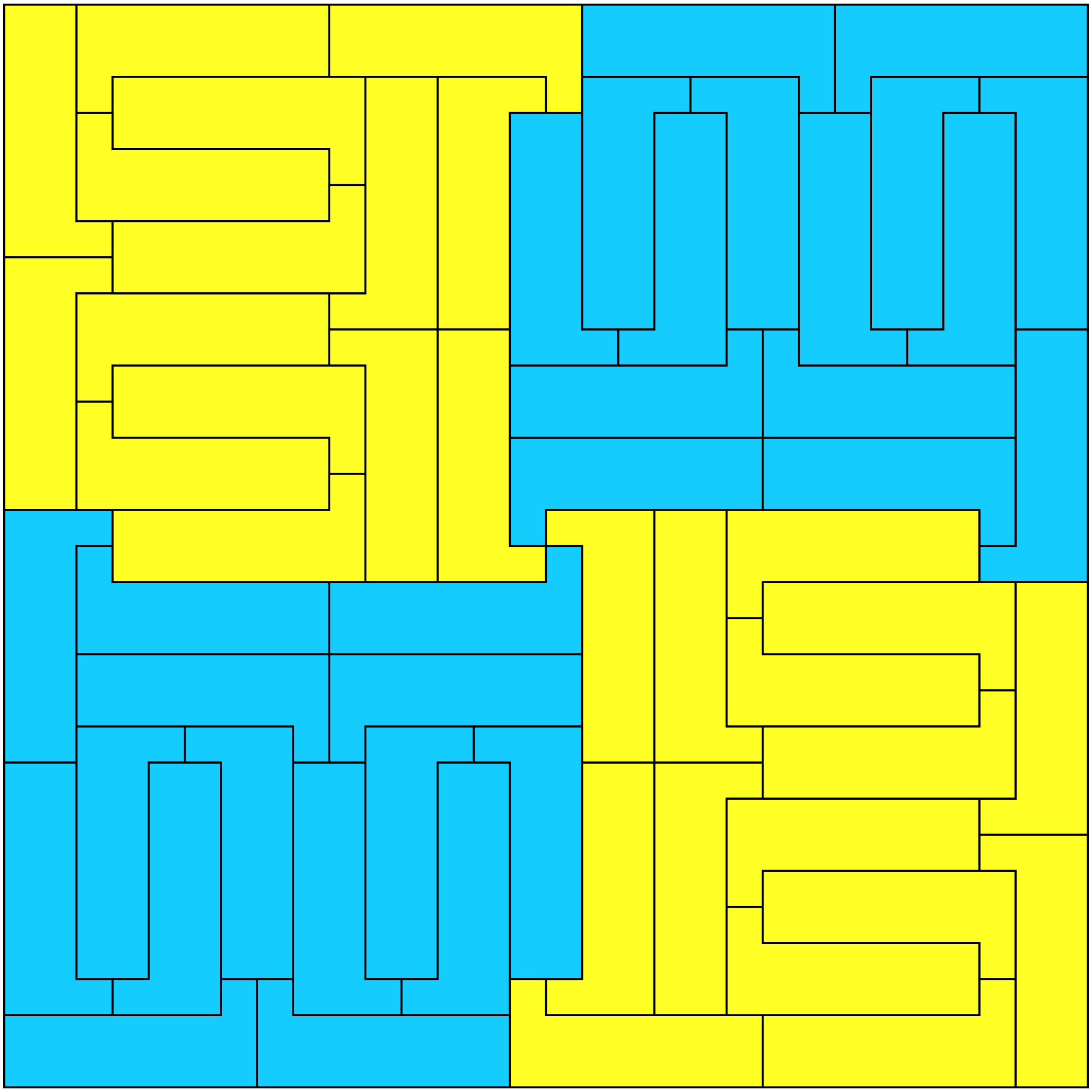


Figure 175: Polyomino with order 60.

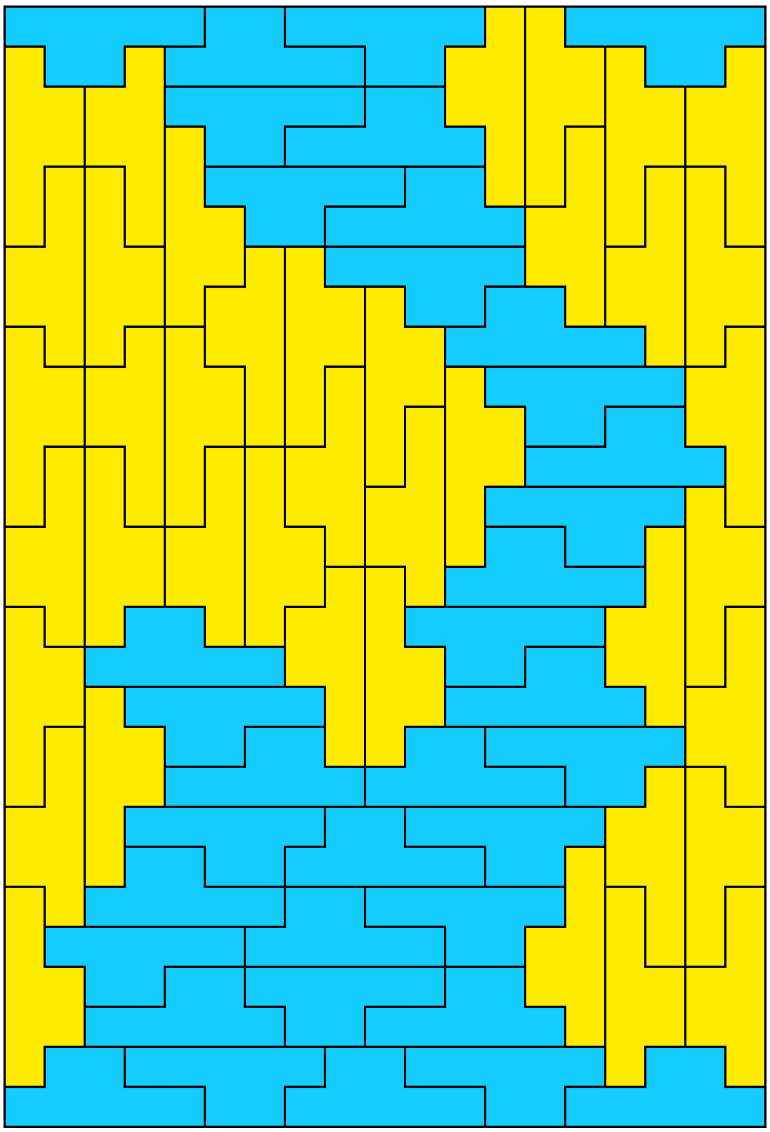


Figure 176: Polyomino with order 76.

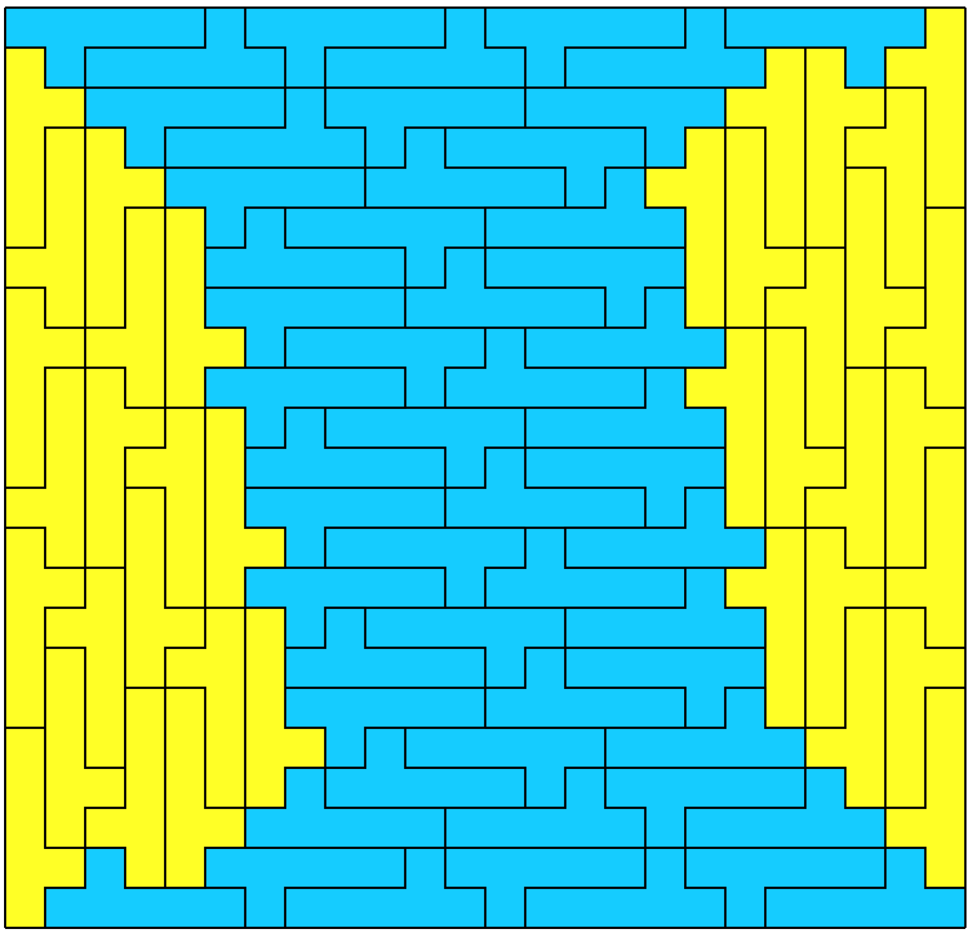


Figure 177: Polyomino with order 92.

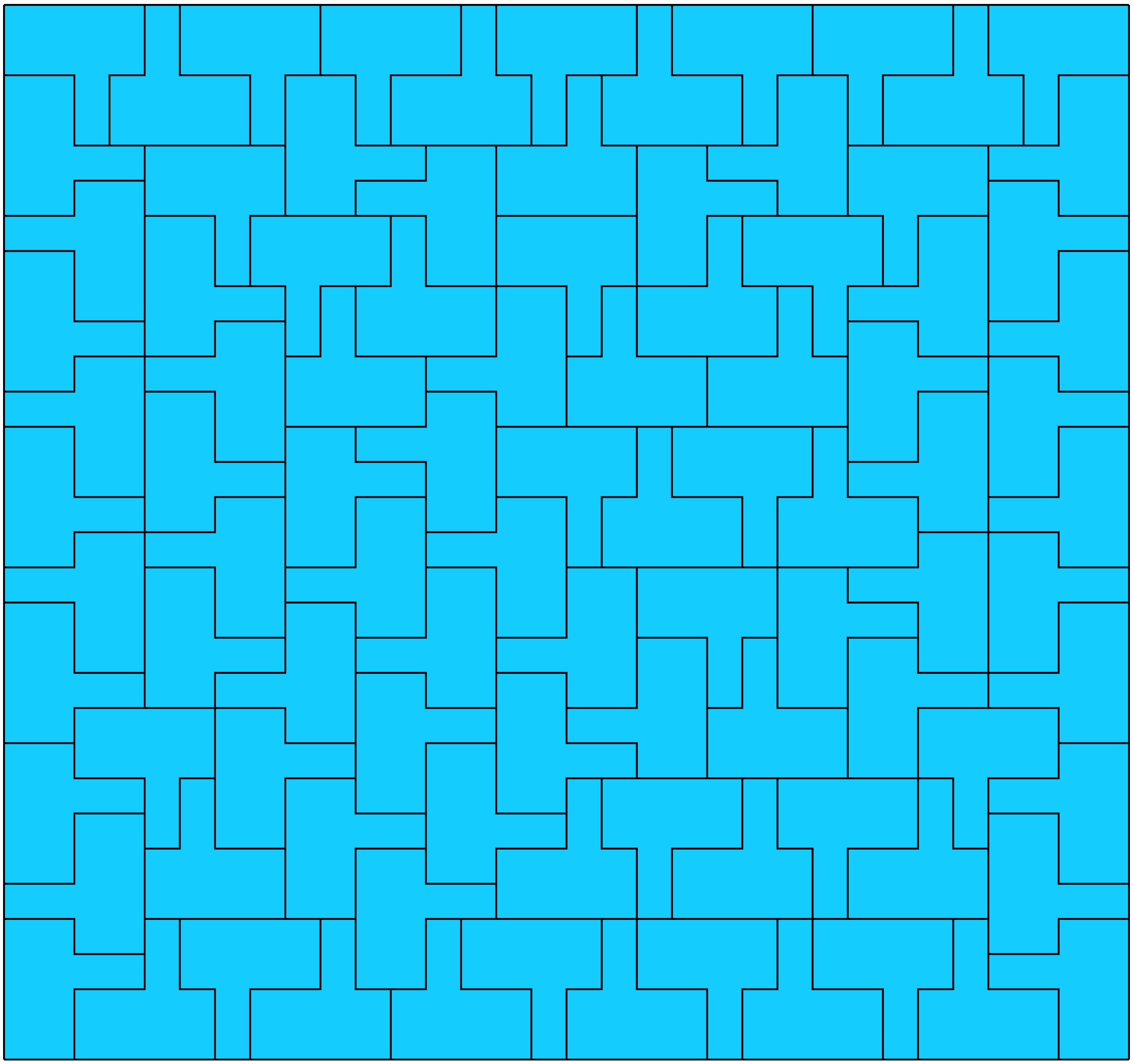


Figure 178: Polyomino with order 96.

Currently, we do not know which other orders greater than 4 are possible, and in particular, we do not know if there are any polyominoes with an order that is odd other than 1 (Winslow, 2018, Open Problem 5).

Only two known polyominoes has order equal to 6 modulo 8: (Reid, http://www.cflmath.com/Polyomino/11omino7_rect.html and http://www.cflmath.com/Polyomino/8omino11_rect.html).

It is easy to come up with polyominoes of order 2: simply take a rectangle with even area, and divide it into two with a lattice curve with 180°-rotational symmetry (such a curve is called **centrosymmetric**).

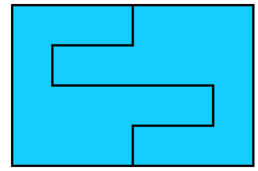


Figure 179: An example of a polyomino with rectangular order 2.

Table 31 shows results for know odd orders. There are also some families, for example odd orders of the form $3(p+2)$ occur for all odd prime p (Reid, 1997).

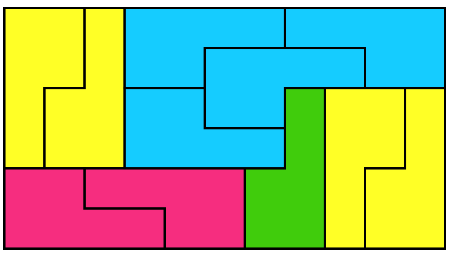


Figure 180: Polyomino with odd order 11.

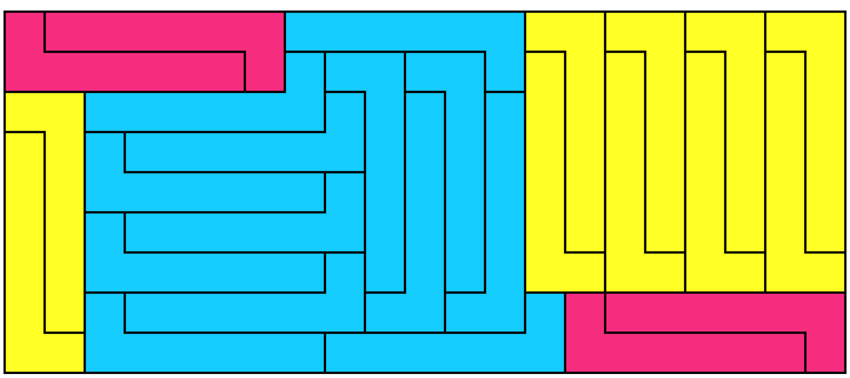


Figure 181: A polyomino with odd order 27.

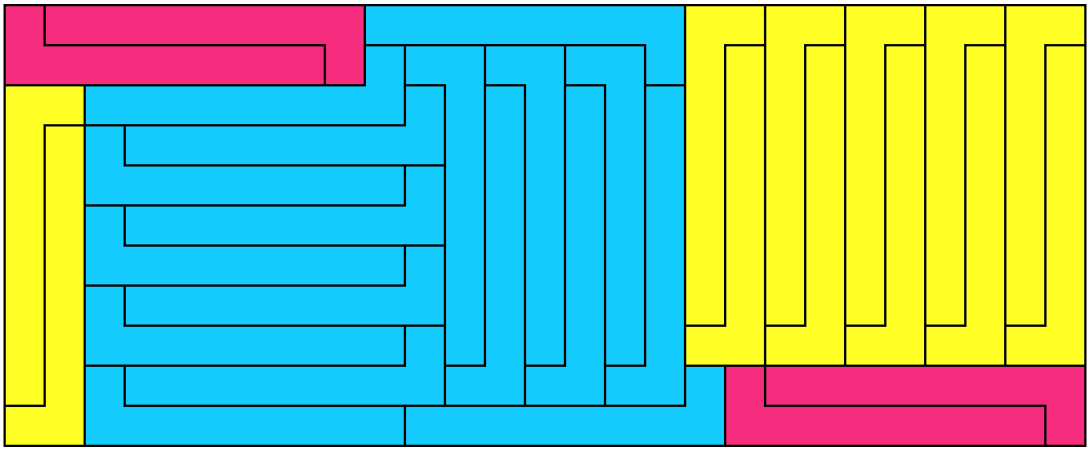


Figure 182: A polyomino with odd order 33.

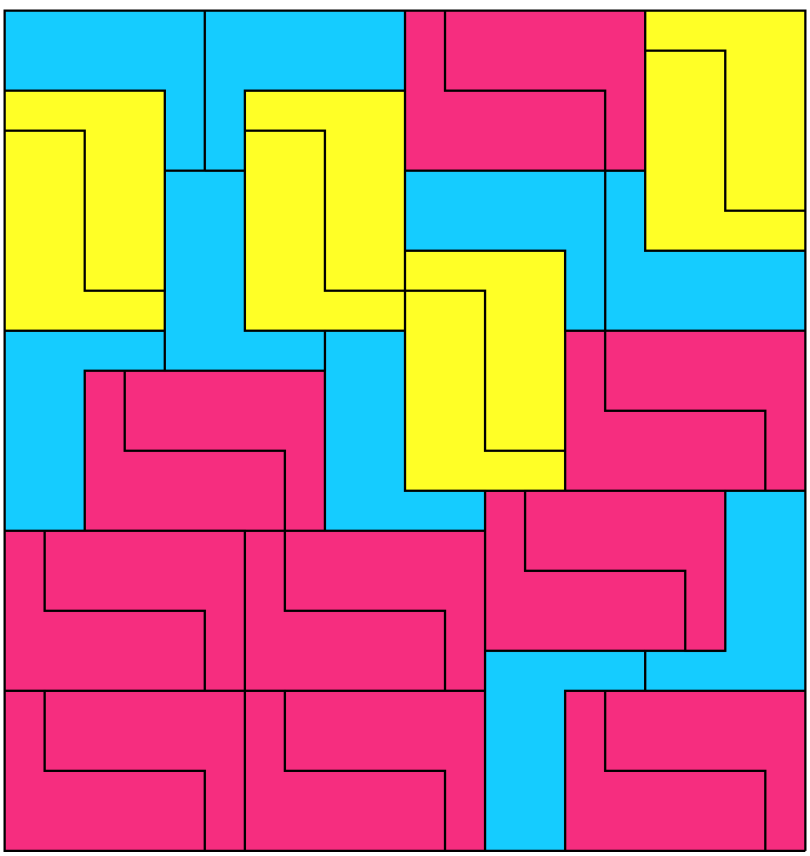


Figure 183: A polyomino with odd order 35.

Table 31: Examples of odd orders. From Golomb (1966), Grekov, Reid (2014).

Odd Order	Tiling
1	All rectangles
11	Figure 180
15	$R(5,9)$ in Figure 203
21	$R(7,15)$ in Figure 209 and 213
27	Figure 181
33	Figure 182
35	Figure 183
45	http://www.cflmath.com/Polyomino/y5_rect.html
	http://www.cflmath.com/Polyomino/7omino4_rect.html

We do know polyominoes cannot have order 3.

Theorem 201 (Stewart and Wormstein (1992)). *If three congruent copies of a connected polyomino P tile a rectangle, then P is itself a rectangle and the tiling can only be one of the two shown in Figure 184.*

[Referenced on page 223]

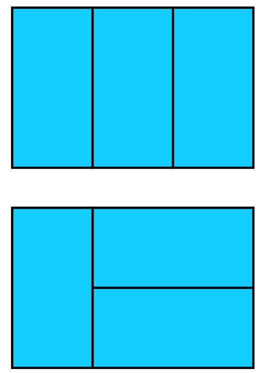


Figure 184: The two ways in which a rectangle can be disected into three congruent shapes.

The proof of this theorem is long, but not very difficult. See the reference for details.

Because rectangles have order 1, this theorem implies no polyomino of order 3 exists.

There are 3 known ways in which polyominoes of order 4 can fit together (Golomb, 1996, p. 89-99):

(1) In a construction with symmetry class **Rot2**.

(2) In a construction with symmetry class **Axis2**.

(3) In a construction with symmetry class **Rot**.

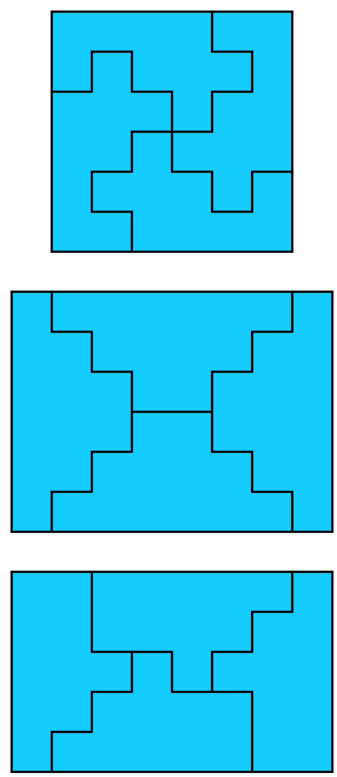


Figure 185: Examples of order 4 polyominoes.

Problem 53.

(1) *Are all of the order 4 polyominoes of this type?*

Most polyominoes that tile rectangles occur as **families**. Currently, there are families of order 2, 4, 8, 10, ...

Here are some known families:

(1) Families of order $4s$ (Golomb, 1996, pp. 102–204).

(2) Family of order 8 (Reid, 1998)[7]. The first three polyominoes is shown in Figure 186.

(3) Families of order $2(a^2+b^2)$ with $\gcd(a,b) = 1$ (Marshall, 1997). This family is constructed from a right triangle with sides a and b. Take b copies of a centrosymmetric curve , and

[7] The first polyomino in this family was discovered by Marshall (1997), but Reid (1998) discovered the family it is part of.

replace side of length a with b copies, and side of length b with a copies. Figures 187-189 shows some examples.

(4) Families with odd order $3n+6$ for odd primes n (Marshall, 1997). Examples are shown in... If n is prime, the $R(n+2, 3n)$ is minimal rectangle with a tiling. It is not know whether the rectangles are minimal for composite n, except for $n = 9, 15, 21$.

6.2.4 The orders of small polyominoes

The orders of small polyominoes is given in Table 32.

6.3 Prime Rectangles

A **prime rectangle** of a polyomino is a rectangle that can be tiled by that polyomino, and cannot be split into two smaller rectangles that can be tiled by that polyomino. A **strong prime rectangle** is a rectangle that can be tiled by the polyomino, and cannot be tiled by smaller rectangles that can be tiled by that polyomino (Reid, 2005, 3.1 – 3.5).[8]

It should be clear that every strong prime rectangle is a prime rectangle. Is the converse true? For sets of polyominoes, it is not. For example, I_3 and I_4 tile $R(5,5)$, but cannot do so with a fault. However, for single polyominoes we do not know.

[8] Some authors use the word "prime rectangle" for what we refer to as a "strong prime rectangle" ((Klarner, 1981)).

Problem 54 (Reid (2005), Question 3.9)**.** *Is there a polyomino with a prime rectangle that is not a strong prime rectangle? This is an open problem.*

Theorem 202 (Reid (2005), Theorem 3.6)**.** *The set of prime rectangles of a polyomino is finite.*

[Not referenced]

See the reference for a proof. For two alternative proofs, see de Bruijn and Klarner (1975).

Theorem 203 (Reid (2005), Proposition 3.10)**.** *A rectangle has only one prime rectangle: itself.*[9]

[9] Mentioned without proof in Klarner (1981).

[Not referenced]

Proof. We need to show that any rectangle $R(m,n) \neq R(p,q)$ that is tileable by a rectangle $P = R(p,q)$ can be split into two rectangles that can each be split into two rectangles, each of which is tileable by P.

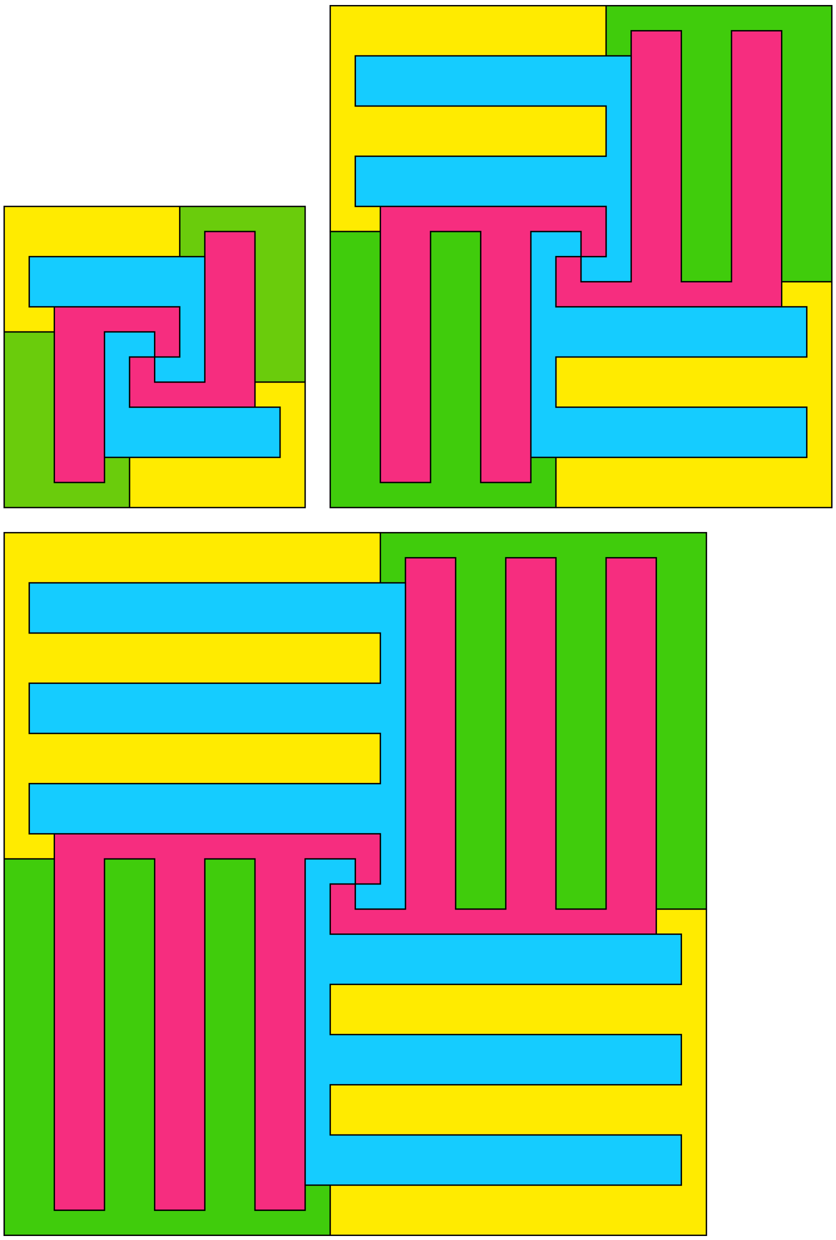


Figure 186: A family of polyominoes of order 8.

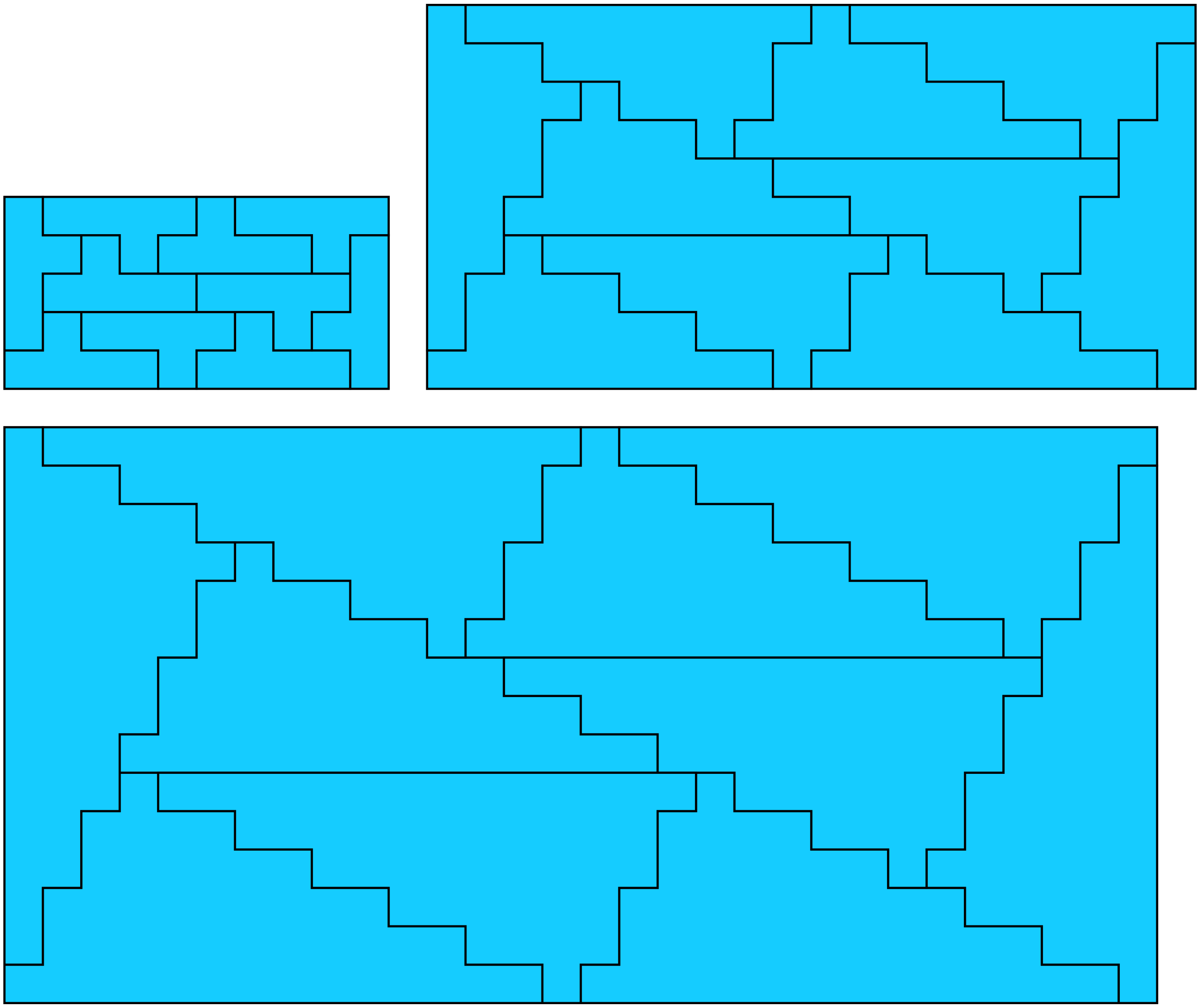


Figure 187: Order 10

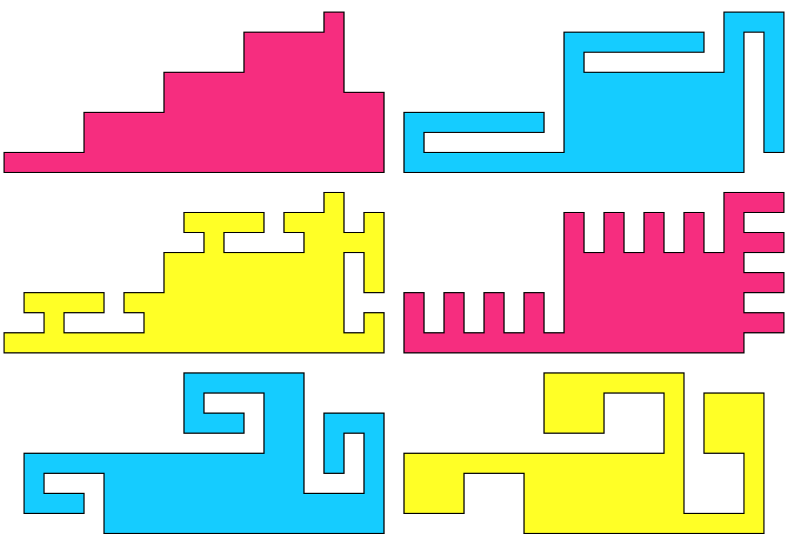


Figure 188: Order 10

Polyomino	O	PR	SO	OO	MR
Monomino	1	1	1	1	2
Domino	1	1	2	1	2
Bar	1	1	3	1	2
Right	2	2	12	15	2
Bar	1	1	4	1	2
Square	1	1	1	1	2
L	2	2	4	∞	2
T	4	1	4	∞	4
I	1	1	5	1	2
L	2	2	20	21	4
P	2	2	20	21	2
Y	10	40	20	45	9
	1	1	6	1	2
	1	1	6	1	2
	2	≥ 8	6	21	?
	2	≥ 1	24	?	?
	2	≥ 1	24	?	?
	2	≥ 1	24	?	?
	2	5	6	11	?
	4	2	24	?	?
	18	≥ 26	96	∞	?
	92[a]	≥ 45	≤ 384	?	?
	1	1	7	1	2
	2	2	28	27	6
	2	≥ 3	28	33	4
	2	≥ 4	28	≤ 57	?
	28	≥ 37	28	45	?
	76[b]	≥ 26	?	≤ 153	?
	1	1	8	1	2
	1	1	2	1	2
	2	≥ 4	2	?	?
	2	≥ 1	2	?	?
	2	≥ 5	8	?	?
	2	≥ 5	8	?	?
	1	1	9	1	2
	1	1	1	1	2
	2	≥ 2	36	≤ 33	?
	2	≥ 4	4	15	?
	4	≥ 15	4	≤ 87	?
	8	≥ 2	12	?	?

Table 32: Table giving order information about polyominoes. Adapted from Grekov. See http://polyominoes.org/rectifiable.

O = Order
PR = Prime Rectangles
SO = Square Order
OO = Odd Order
MR = Minimum Reptile
[a]Dahlke (1989b) [b]Dahlke (1989a)

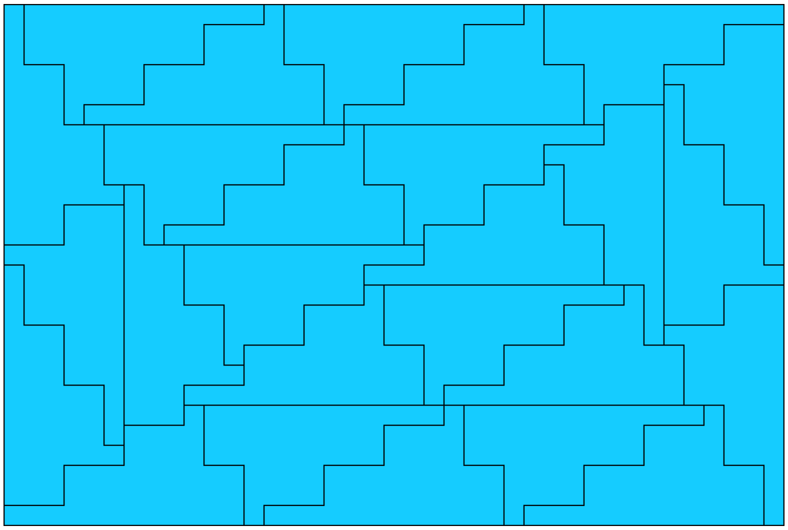


Figure 189: Order 26

From Theorem 160 we know that one of the following is true:

(1) $p \mid m$ and $q \mid n$

(2) $pq \mid m$ and $n = px + qy$ for $x, y > 0$.

In the first case, we can divide the rectangle into either

- $R(m-p, n)$ and $R(p, n)$ if $m > p$
- $R(m, n-q)$ and $R(m, q)$ if $n > q$

In the second case, we can divide the rectangle in $R(m, px)$ and $R(m, qy)$, two rectangles of case (1).

So no other rectangle other than $R(p, q)$ can be a prime of $R(p, q)$.

□

6.3.1 *Trominoes*

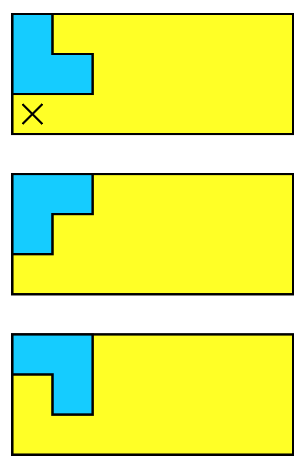


Figure 190: The three ways a tromino can fit into the top left corner of a rectangle.

Theorem 204 (Chu and Johnsonbaugh (1985)). *The right tromino tiles a rectangle iff* $3|mn$*, and when one side is* 3 *the other is even.*

[Referenced on page 190]

Proof. (The proof follows the outline in Ash and Golomb (2004), p. 48.)

(1) $R(3m, 2n)$ divides into m columns and n rows of $R(3, 2)$, which is tileable by right trominoes.

(2) $R(6k, 2n+3)$ can be split into one $R(6k, 3)$ and several $R(6k, 2n)$ rectangles. The first is tileable with $3k$ copies of $R(2, 3)$, and the second by (1).

(3) $R(9+6k, 2n+5)$ can be split into four rectangles: $R(9,5)$ (see Fig 203), $R(6k, 2n+5)$ (tileable by (2)), $(9, 2n)$ (tileable by (1)) and $R(6k, n2)$ (tileable by (1)).

(4) $R(3, 2n+1)$ is not tileable. There are only three ways to place a tromino in the top left corner. Only two of them can work; for both these cases the only tiling option is to complete the $R(3,2)$ rectangle. This reduces the tiling to that of $R(3, 2n-1)$. We continue this, until a single row of three cells is left, which is untileable.

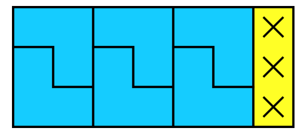


Figure 191: Trying to tile $R(3, 2n+1)$ with right trominoes always result in a column that cannot be tiled.

□

6.3.2 Tetrominoes

Theorem 205 (Martin (1991), p. 50). *The skew tetromino does not tile any rectangle.*

[Not referenced]

Proof. The proof follows directly from Thereom 186.

Here is an alternative proof given in Martin (1991): The polyomino can be placed in the corner in one of two ways, which are equivalent. The placement produces two notches which can fit the polyomino only one way each, so that both neighbors are determined. This produces new notches, which eventually result in a untileable cell in the adjacent corner of the rectangle. □

Theorem 206 (Golomb and Klarner (1963)). *The L-tetromino tiles $R(m, n)$ iff $8 \mid mn$ and $m, n > 1$.*

[Not referenced]

Proof. The L-tetromino satisfies the conditions of Theorem 179, and hence it is even. Therefor, the area must be divisable by 8. The proof that all these rectangles have tilings uses the same ideas as used in Theorem 204. Figure 205 shows a tiling for $R(8,3)$. □

We next want to prove that if a rectangle is tileable by T-tetrominoes, both sides are divisible by four. To do this, we will show that any tiling of the quadrant satisfy certain properties, and if we stack rectangles together to form a quadrant tiling, these properties can only be satisfied if both sides are divisible by 4.

We introduce some terminology from (Walkup, 1965, Definition 1). An edge of the quadrant is called a **cut** if it coincides with an outer edge of some T-tetromino in every tiling of the quadrant. A vertex in the quadrant is called **cornerless** if it does not coincide with an outside vertex of a T-tetromino in any tiling of the quadrant. A vertex is a **type-A vertex** if it is congruent to $(0,0)$ or $(2,2)$ modulo 4. A vertex is a **type-B vertex** if it is congruent to $(0,2)$ or $(2,0)$ modulo 4.

Theorem 207 (Walkup (1965), Lemma 1)**.** *In a tiling of the quadrant by T-tetrominoes, every type-B vertex is cornerless, and every edge incident with a type-A vertex is a cut.*

[Referenced on page 192]

Proof. For integers $k \geq 0$, let $P(k)$ be the proposition that the theorem holds for all type-A and type-B vertices on or below the line $x + y = 4k$.

$P(0)$ is true, because the two edges incident on the origin are cuts. We will prove that $P(k)$ implies $P(k+1)$, and will use the Figure 192 as reference, which shows the situation for $k = 2$. The cuts and cornerless points required for $P(k)$ are shown by heavy lines and dots.

(1) We show that edges a and b, and their translates, are cuts.

If a tiling of the quadrant contains the tetromino $1-2-3-4$, it will also contain $8-10-11-12$, and so all upwards translates of $1-2-3-4$, because no other arrangement can cover cells 8 and 9. But this leads to two cells $(13,14)$ at the y-axis that cannot be covered. Therefor, the tiling cannot contain $1-2-3-4$. By symmetry, it can also not contain $1-5-6-7$.

There are four other ways to cover 1; all of them has a and b as outer edges, and therefor a and b are cuts, and so are their translates. These are marked in Figure 2.

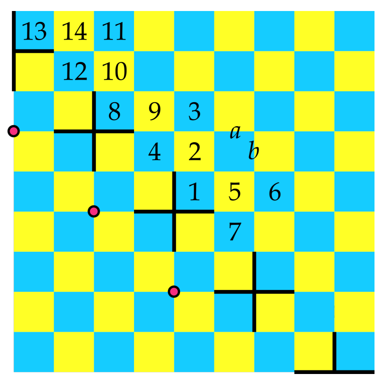


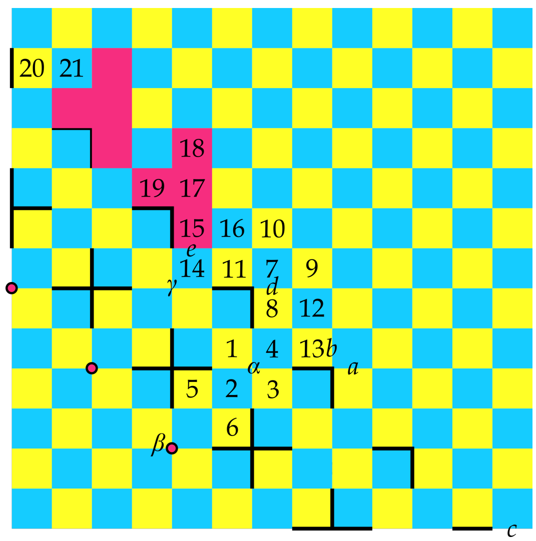
Figure 192:

(2) We now show that point α is cornerless. Suppose, to the contrary, that there is a tiling containing a T-tetromino having α as an outside corner.

In this tetromino the cell forming the outside corner at α cannot be 1, 2, or 3 because of the nearby cut segments. Consequently, either $1-2-3-5$ or $1-2-3-6$ is a tetromino of the supposed dissection.

But no tetromino could then contain square 6 or 5 respectively, as A is a cornerless point. This proves that α is cornerless. By the same or similar arguments all translates of α, either inside the quadrant or on the axes, are cornerless.

(3) Finally, we show edges a and b, and their translates, are cuts.



By symmetry, it is enough to show a is a cut, and since c is a cut, it is enough to show that if an edge a is a cut, so is its translate d.

Suppose, a is a cut, and suppose, for a contradiction, that cells 7 and 8 lie in the same tetromino. The tetromino cannot contain 4, since α is cornerless. It must contain two of the three cells 9, 10, and 11. If it contained 9, there would be no way to cover 12 and 13. The tetromino must thus be $7 - 8 - 10 - 11$. The cells 14 and 15 must be covered by different tetrominoes, since γ is cornerless. Then the only way to cover 15 and 16 is for $15 - 17 - 18 - 19$ to be part of the tiling. Then upwards translates of $15 - 17 - 18 - 19$ must be part of the tiling, which means that there is no way to cover 20 and 21, a contradiction. Therefor, 7 and 8 must lie in different tetrominoes and so d is a cut.

We have shown a and b and their translates are cuts.

Taken together, we have shown that $P(k)$ implies $P(k+1)$, and the theorem follows from induction. □

Theorem 208 (Walkup (1965), Theorem 1). *The T-tetromino tiles $R(m,n)$ iff $4 \mid m$ and $4 \mid n$.*

[Referenced on page 207]

Proof. *If.* The T-tetromino tiles $R(4,4)$; these can be stacked to form any rectangle $R(m,n) = R(4m', 4n')$.

Only if. We cannot have m congruent to 2 module 4, because this puts a corner of R at a type-B vertex, which is impossible since type-B points are cornerless (Theorem 207). Moreover, m cannot be congruent to 1 or 3 module 4. If the edge of the rectangle is placed at either P or Q, there is no way to cover cells 22 or 23. Thus, m and by symmetry, must be congruent 0 modulo 4. □

Problem 55 (Walkup (1965), Definition 2, Theorem 2)**.** *Let* $R = R(m,n)$ *be a rectangle with a tiling by T-tetrominoes. A block is a* 2×2 *square whose vertices have even coordinates. A chain of R is any minimal subset of R that is both a union of blocks and a union of T-tetrominoes from the tiling.*

Show that if we color every other block of R in the checkerboard fashion, then each tetromino has three cells in one block and one in an adjacent block. Every chain consists out of an even number of blocks, which may be cyclically ordered in such a way that the blocks are alternately colored black and white, and the T-tetrominoes of the chain contain three cells of the one block and one cell of the succeeding block.

6.3.3 *Pentominoes*

We state a useful result for V-like n-ominoes with one leg length 3 and the other leg length $n-2$. We use the notation V_n for these polyominoes (Saxton Jr, 2015).

Theorem 209 (Saxton Jr (2015), Theorem 2.3, p. 10)**.** *For* $n \leq 5$*,* V_n *does not tile a quadrant.*

[Referenced on pages 193 and 195]

The proof is long and tedious with many cases. See the reference for details.

Theorem 210 (Cibulis and Liu (2001))**.** *The F-, S-, T-, U-, V-, W-, X-, and Z-pentominoes do not tile a rectangle.*

[Not referenced]

Proof. The X-pentomino does not cover any corner of its hull, so it cannot tile a rectangle 181.

Placing the F-, T- and Z-pentominoes in any position in a corner leaves one cell that cannot be tiled.

The U-pentomino can cover the corner in only one way without cutting off any cells. But then the gap in the U can be covered only one way, which leaves one cell that cannot be covered.

The S- and W-pentominoes are not rectifiable by Theorem 186.

The V-pentomino cannot tile a quadrant (Theorem 209) and thus not a rectangle (Theorem 133).

An alternative proof from Cibulis and Liu (2001): The V-pentomino can be placed in the corner in two ways. The first creates an inaccessible pair. The second, forces a second to be placed as shown (otherwise it is not tileable for the same reason as before), and a third. This configuration leads to an inaccessible pair. □

Theorem 211. *The L-pentomino tiles:*

(1) $R(5m, 2n)$

(2) $R(10m, 2)$, $R(10m, n+4)$

(3) $R(15m, 7+2n)$

[Not referenced]

Theorem 212. *The P-pentomino tiles these and only these rectangles:*

(1) $R(5m, 2n)$

(2) $R(10m, 2)$, $R(10m, n+4)$

(3) $R(15m, 7+2n)$

[Not referenced]

Proof. Figure 209 gives tilings of $R(2,5)$ and $R(15,7)$. From this we can construct $R(5m, 2n)$ which proves (1). In particular, we can construct $R(15,2)$, so together with $R(15,7)$ we can construct all $R(15, 7+2k)$, which proves (3). From (1) we can construct $R(10,5)$ and $R(10,2)$, putting these together proves (2).

Since $5|mn$ (Theorem 1), 5 divides either m or n. The only remaining rectangles we have not given tilings for are $R(1, 5m)$, $R(5, 2n+1)$, and $R(10,3)$. We prove now these are impossible.

The pentomino cannot fit $R(5,1)$, so $m, n > 1$.

Any rectangle that fits k polyominoes must also fit k $R(2,2)$ squares.

$R(5, 2n+1)$ fits $\left\lfloor \frac{5}{2} \right\rfloor \left\lfloor \frac{2n+1}{2} \right\rfloor = 2n$ squares (Theorem 172), but $2n+1$ are required; therefor no tiling of $R(5, 2n+1)$ is possible.

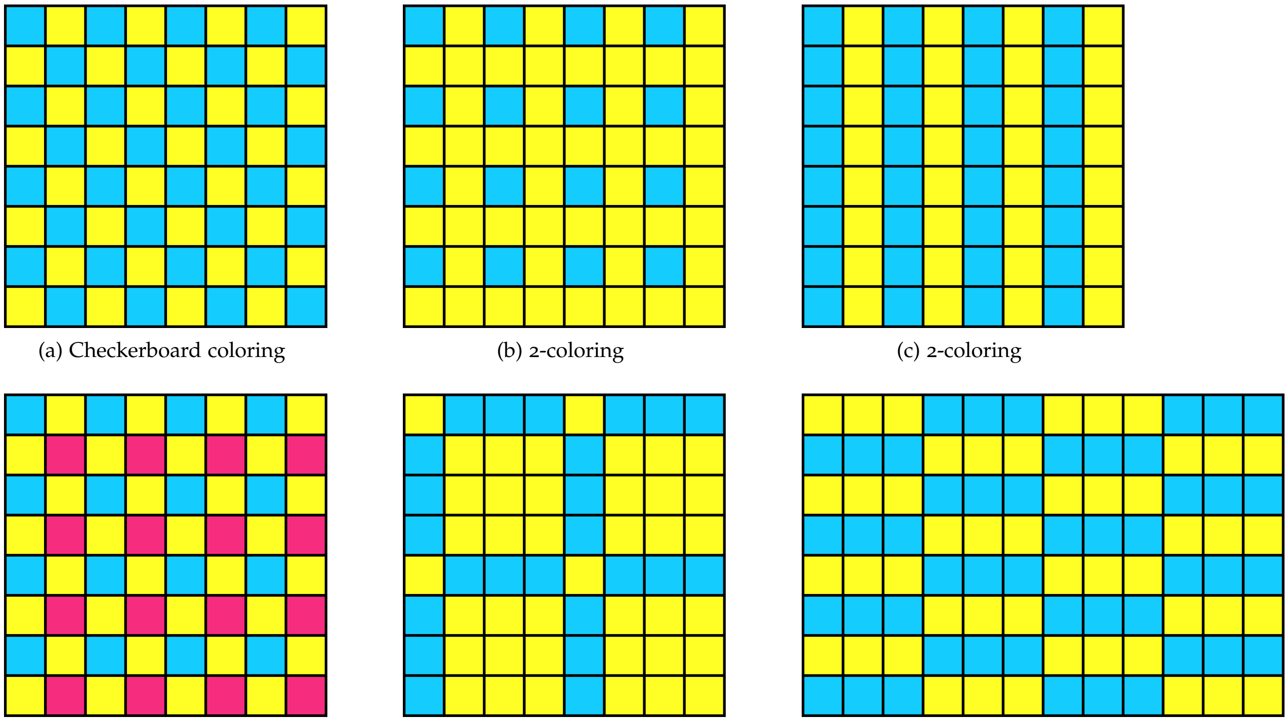
Similarly, $R(10,3)$ fits only 5 squares, but 6 are required. So $R(10,3)$ cannot be tiled. □

Theorem 213 (Sillke (1992)[10]). *The complete list of 40 prime-rectangles of the Y-pentomino is given in Table 34.*

[Not referenced]

[10] According to Reid (http://www.cflmath.com/Polyomino/y5_rect.html) the first complete list was published in Fogel et al. (2001), but I could not locate the reference. The complete list was identified before then by Sillke (1992). Reid (2005, Example 5.2) gives an overview of the publishing history of the prime rectangles.

Proof. See Reid (2005, Example 5.2) for an outline. □



(a) Checkerboard coloring (b) 2-coloring (c) 2-coloring

(d) 3-coloring (e) 2-coloring (f) 2-coloring

Figure 193: Various coloring used for proofs in this section.

6.3.4 *Hexominoes*

Hexomino	Reason	Hexomino	Reason	Hexomino	Reason	Hexomino	Reason
A	✓	J	✓	R	✓	X	Thm. 181
C	Thm. 189	K	×	Long S	Thm. 186	Italic X	Thm. 181
D	✓	L	✓	Long T	×	High Y	✓
E	×	M	×	Short T	××	Low Y	Thm. 189
High F	×	Long N	Thm. 186	U	✓	Short Z	Thm. 186
Low F	×	Short N	Thm. 186	V	Thm. 209	Tall Z	×
G	Thm. 188	O	✓	Wa	××	High 4	×
H	×	P	✓	Wb	Thm. 186	Low 4	×
I	✓	Q	Thm. 186	Wc	×		

Table 33: Summary of proofs for pentominoes that do not tile rectangles.

Proving which hexominoes cannot tile rectangles is a bit tedious. In most cases we find an inaccessible cell within two steps, in other cases we can use one of the theorems we to show the polyomino is untilable. This is summarized in Table 33.

Theorem 214 (Reid[11]). *If the U-hexomino tiles $R(m, n)$, then*

[11] http://www.cflmath.com/Polyomino/f6_rect.html).
Note that Reid calls this hexomino the F-hexomino.

(1) $12 \mid mn$

(2) $4 \mid m$ *or* $4 \mid n$

.

[Not referenced]

Proof. The U-hexomino is unbalanced, so the first part follows from Theorem 179.

This implies that all rectangles are of the form $R(4m, n)$ or $R(2m + 2, 2n + 2)$. We prove that the $R(2m + 2, 2n + 2)$ rectangle cannot be tiled: Color the rectangle such that (x, y) is black if x and y are both even. No matter how the polyomino is placed, it covers an even number of black cells. However, $R(2m + 2, 2n + 2)$ has an odd number of black cells. Therefor, it cannot be tiled. □

Theorem 215 (Reid[12]). *If the J-hexomino tiles $R(m, n)$, then $4 \mid m$ or $4 \mid n$.*

[12] http://www.cflmath.com/Polyomino/j6_rect.html

[Not referenced]

Proof. The polyomino always tiles in pairs, as shown in Figure 194, so we only need to show the tiling capability of these shapes. The area is therefor divisible by 12, and so it suffices to prove that we cannot tile $R(4m + 2, 4n + 2)$.

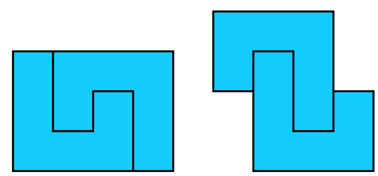


Figure 194: J-hexominoes always tile in pairs.

Apply this coloring: a cell is black if and only if x or y, but not both, are divisible by 4 (Figure 193(e)). No matter how we place one of the pairs, the pair always cover an odd number of black cells. $R(4m + 2, 4n + 2)$ has an even number of black cells; however, it must be tiled by an odd number of pairs, which will cover an odd number of black cells, which is a contradiction. Therefor, a tiling of $R(4m + 2, 4n + 2)$ is impossible, and so one side must be divisible by 4. □

Theorem 216 (Reid[13]). *If the A-hexomino tiles $R(m, n)$, then $4 \mid m$ or $4 \mid n$.*

[13] http://www.cflmath.com/Polyomino/a6_rect.html

[Not referenced]

Proof. The A-hexomino is unbalanced with an even number if cells, and so is even (Theorem 179). Thus the area is divisible by 12, and it suffices to show that we cannot tile $R(4m + 2, 4n + 2)$.

Apply this coloring: a cell is black if and only if x and y are both even (Figure 193(b)). No matter how we place the polyomino, it always cover an odd number of black cells. $R(4m + 2, 4n + 2)$ has an even number of black cells; however, it must be tiled by an odd number of pairs, which will cover an odd number of black cells, which is a contradiction. Therefor, a tiling of $R(4m+2, 4n+2)$ is impossible, and so one side must be divisible by 4. □

Theorem 217 (Reid[14]). *If D-hexomino tiles* $R(m,n)$*, then* $6 \mid m$ *or* $6 \mid n$*.*

[14] http://www.cflmath.com/Polyomino/d6_rect.html

[Not referenced]

Proof. Apply this coloring: a cell is black if and only if $x + \lfloor \frac{y}{3} \rfloor$ is even (Figure 193(f)). No matter how we place the polyomino, it always cover the same number of black and white squares. For rectangles of the forms $R(6m+2, 6n+3)$ and $R(6m+4, 6n+3)$, we can be colored so that the number of black and white cells are different; which means they cannot be tiled. The only other possibility is $R(6m, n)$ or $R(m, 6n)$, which means 6 divides on of the sides. □

Theorem 218 (Reid[15];Reid (2005), Theorem 5.12). *If D-hexomino tiles* $R(m,n)$*, then* $4 \mid m$ *or* $4 \mid n$*.*

[15] http://www.cflmath.com/Polyomino/d6_rect.html

[Not referenced]

The proof is not very difficult, but requires some concepts that we have not covered. See the second reference for details.

Theorem 219 (Reid[16]; Reid (2003), Theorem 5.4). *If a G-hexomino tiles* $R(m,n)$*, then* $4 \mid m$ *or* $4 \mid n$*.*

[16] http://www.cflmath.com/Polyomino/g6_rect.html

[Not referenced]

As with the previous theorem the proof requires concepts we have not covered. See the second reference for details.

Theorem 220 (Reid[17]). *If the Y-hexomino tiles a rectangle with an odd side, then the other side is divisible by 8.*

[17] http://www.cflmath.com/Polyomino/y6_rect.html

[Not referenced]

Proof. $R(2m+1, 8n+k)$ is not tileable for odd k, since at least one side must be even (Theorem 1). If $R(2m+1, 8n+2)$ is tileable, then so is $R(2m+1, 8n'+4)$; if $R(2m+1, 8n+6)$ is tileable, then so is $R(2m+1, 8n'+2)$., then so is $R(2m+1, 8n''+4)$.

Therefor, it is enough to show $R(2m+1, 8n+4)$ is not tileable for any integers m and n.

Consider the numbering

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are both even} \\ -1 & \text{if } x \text{ and } y \text{ are both odd} \\ 0 & \text{otherwise.} \end{cases}$$

No matter how the tile is placed, it covers ± 2, or $2 \pmod 4$. $R(2n+1, 8n+4)$ also covers a total that is $2 \pmod 4$. However, it would require an even number of tiles, which would cover a total divisible by 4; a contradiction.

Therefor, $R(2m+1, 8n+4)$ is not tileable by Y-hexominoes, and so neither is $R(2m+1, 8n+k)$ for any k not divisible by 8.

Therefor, if $R(2m+1, n)$ is tileable by Y-hexominoes, n is divisible by 8. □

6.3.5 *Other polyominoes*

Theorem 221 (Klarner (1969), Theorem 6)**.** *The P-octomino tiles $R(m,n)$ iff $m,n > 3$ and $16 \mid mn$.*

[Not referenced]

Proof. *If.* If $m,n > 3$ and $16 \mid mn$, then $R(m,n)$ can be cut into rectangles with dimensions $R(4,4)$, $R(5,16)$, $R(6,8)$, and $R(7,16)$, which are all tileable as shown in Figure 195 (See Problem 56).

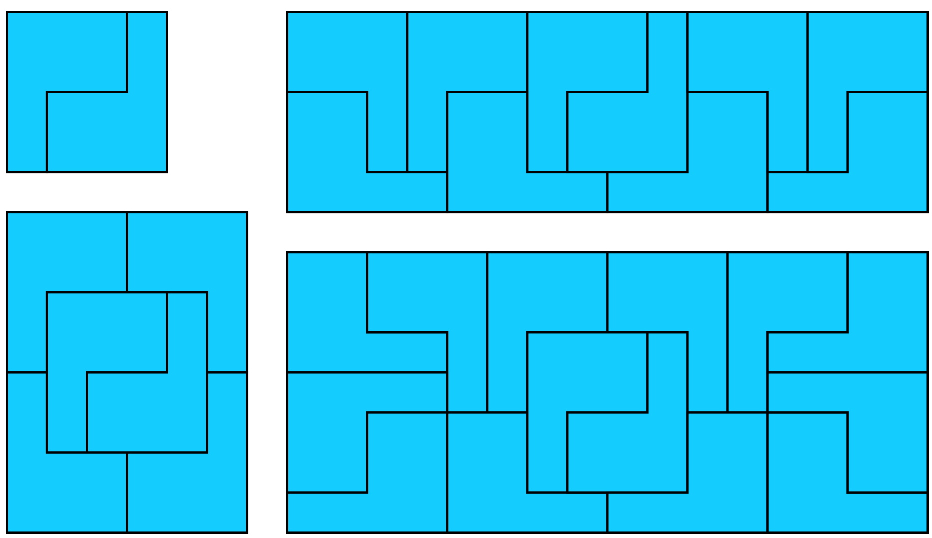


Figure 195: The prime rectangles of the P-octomino.

Only if. If $R(m,n)$ is tileable, it is obvious $m,n > 3$. To show that mn is divisble by 16, we will show the P-octomino is even.

Since 8 divides mn, 4 divides some side, say m. We apply the coloring in shown in Figure 196 on the rectangle. The polyomino is of type (a,b,c) if it covers a amber cells, b blue cells and c cherry cells.

Let $k_{1\ldots6}$ denote the number of tiles in the tiling with types $(3,4,1)$, $(1,3,4)$, $(4,2,2)$, $(2,6,0)$, $(2,2,4)$, $(0,6,2)$ respectively. It is easy to check no other types are possible.

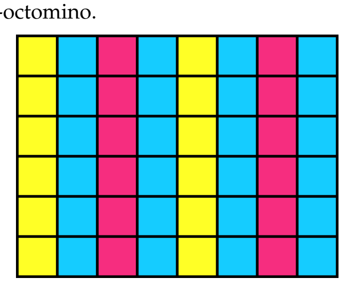


Figure 196: $F_{0,3}$, a 3-coloring.

Since the total number of blue and cherry cells are equal, we have

$$3k_1 + k_2 + 4k_3 + 2k_4 + 2k_5 = k_1 + 3k_2 + 2k_3 + 4k_5 + 2k_6,$$

from which we get

$$k_1 + k_3 + k_4 = k_2 + k_5 + k_6.$$

The total number of tiles is $k_1 + k_2 + \cdots k_6 = 2(k_1 + k_3 + k_4)$, which means the total is even. Hence the P-octomino is even, and hence any rectangle $R(m, n)$ that is tileable by this polyomino must have mn divisible by 16. $\square$

Problem 56. *Show that any rectangle $R(m, n)$ with mn divisible by 16 and $m, n > 3$, can be divided into the four rectangles $R(4, 4)$, $R(5, 16)$, $R(6, 8)$, and $R(7, 16)$.*

6.3.6 Prime Rectangles of Small Polyominoes

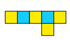
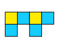
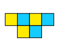
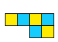
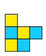
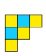
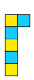
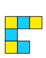
Polyomino	Prime Rectangles							
	3×2	9×5						
L	4×2	8×3						
T	4×4							
P	5×2	15×7						
L	5×2	15×7						
Y	10×5	20×9	30×9	45×9	55×9	14×10	16×10	23×10
	27×10	20×11	30×11	35×11	45×11	50×12	55×12	60×12
	65×12	70×12	75×12	80×12	85×12	90×12	95×12	20×13
	30×13	35×13	45×13	15×14	15×15	16×15	17×15	19×15
	21×15	22×15	23×15	20×17	25×17	25×18	35×18	25×22
	23×24	24×29	24×35	24×41	24×47	24×53	24×59	24×63
	24×65	24×71	24×77	24×83	24×89	24×95	24×101	24×102
	24×103	24×107	24×108	24×113	24×114	24×119	24×120	30×64
	30×68	30×72	30×80	30×88	30×92	30×96	30×100	30×104
	30×106	30×108	30×112	30×116	30×120	30×124	$\cdots$	32×36
	32×42	32×48	32×54	32×60	32×66	$\cdots$	48×48	$\cdots$
	$\cdots$							
	3×4							
	4×6	5×12						
	2×7	11×49	13×35	19×21				
	9×12	9×20	9×28	12×13	12×14	12×17	12×19	12×21
	12×24	12×25	12×29	15×28	15×32	15×36	15×40	15×44
	15×48	15×52	16×18	16×27	16×30	16×33	16×39	16×42
	20×21	20×24						
	3×4							
	2×6	7×12	8×15	9×14	9×16	9×34	10×15	11×18
	3×4							

Table 34: Prime Rectangles of polyominoes

6.4 *Fault-free tilings*

Recall that a *fault* in a tiling is a line on the grid (horizontal and vertical) that goes through the figure and is not crossed by any tile.

We are mostly interested in fault-free tilings because a fault-free tiling can not be broken down into two (or more) tilings by smaller rectangles.

Any prime rectangle is fault-free by definition. To find all possible rectangles of a polyomino that has fault-free tilings, we make use of following

- The prime rectangles we looked at in the previous section.
- The theorems describing which rectangles have fault-free tilings by rectangles (described later in this section).
- Tiling **extensions**: systematic ways of building bigger rectangles from smaller ones. We will see several ways that fault-free rectangles can be extended.
- Finding more **basic** tilings of rectangles not covered by any of the above or proving that they don't exist. Once we have a basic rectangle, we also apply the three methods above to extend the set of rectangles with fault-free tilings.

Polyomino	Fault-Free Rectangles			
	$m \times n,$	$2 \mid mn,$	$m, n > 5,$	$(m, n) \neq (6, 6)$
	$m \times n,$	$3 \mid mn,$	$m, n > 3$	
I	$m \times n,$	$4 \mid (p, q),$	$p, q > 8$	
T	$4m \times 4n$			
L	$4k \times 4k$	$(2+4k)k$	$(3+4k) \times 8k$	$(5+4k) \times 8k$
P	$m \times n$	$5 \mid mn$	if $m = 2$ then $n = 5$,	if $n = 2$ then $m = 5$
L	?			
Y	?			

Table 35: Fault-free Rectangles of polyominoes

6.4.1 *Rectangles*

Theorem 222. *Tilings of a rectangle by a square cannot be fault-free.*

[Referenced on page 207]

Proof. Suppose the square has side k. Then the only rectangles tileable are $R(mk, nk)$ for some integers m and n (Theorem 154). A tiling is given by putting m squares in each of n rows. By Theorem 3 this tiling is unique, and since it has a fault, no fault-free tilings of the rectangle exists. □

Theorem 223 (Robinson (1982), Theorem 4)**.** *Let* $1 < m < n$ *and* $\gcd(m, n) = 1$*. The only rectangles with fault-free tilings of a* $m \times n$ *polyomino are these:*

(1) $(kmn + pm + qn) \times lmn$, *with* $k \geq 1$, $l \geq 3$, $0 < p < n$, $0 < q < m$ *(Figures 197 and 198)*

(2) $(kmn + pm) \times (lmn + qn)$, *with* $k, l \geq 2$, $0 < p < n$, $0 < q < m$ *(Figure 199)*

(3) $(kmn + pm) \times (lmn)$, *with* $k \geq 2$, $l \geq 3$, $0 < p < n$ *(Figures 199 and 200)*

(4) $(kmn) \times (lmn + qn)$, *with* $k \geq 3$, $l \geq 2$, $0 < q < m$ *(Figure 198)*

(5) $(kmn) \times (lmn)$, *with* $k, l \geq 3$ *(Figure 198)*

[Referenced on page 214]

To get some intuition of the this theorem, a useful exercise is to construct one of the minimal rectangles in Figures 197 –200 for a given tile, and see if you can find the ways to extend it (without referring to the figure). After this, you will be able to be able to see more easily how the diagrams can be stretched for different tilings.

The following theorem is equivalent to the above, but often easier to work with.

Theorem 224 (Graham (1981), Theorem p.125)**.** *Let* $1 \leq m < n$ *and* $\gcd(m, n) = 1$*. Then a rectangle* p, q *has a fault free tiling if and only if:*

(1) *Each of* m *and* n *divides* p *or* q*,*

(2) p *and* q *can each be expressed as* $xm + yn$ *in at least two ways, with* $x, y > 0$*, and*

(3) *for* $m = 1$ *and* $n = 2$*,* $p, q \neq 6$*.*

[Referenced on pages 205 and 207]

We state a special case for bars:

Theorem 225. *A rectangle* $R(p, q)$ *has a fault-free tiling by bars* $R(1, m)$ *if and only if both the following hold*

(1) $m \mid p$ *or* $m \mid q$

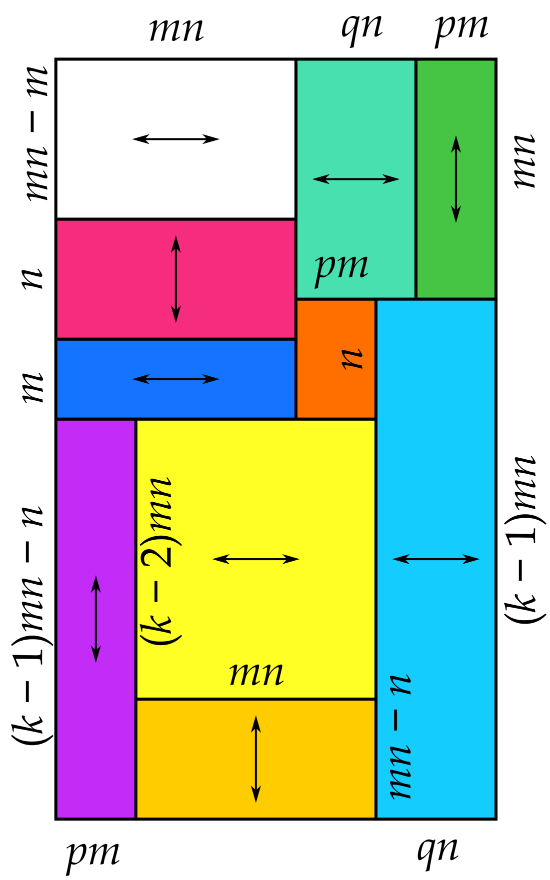


Figure 197: $kmn \times (mn + pm + qn)$, $k \geq 3$, $0 < p < n$, $0 < q < m$

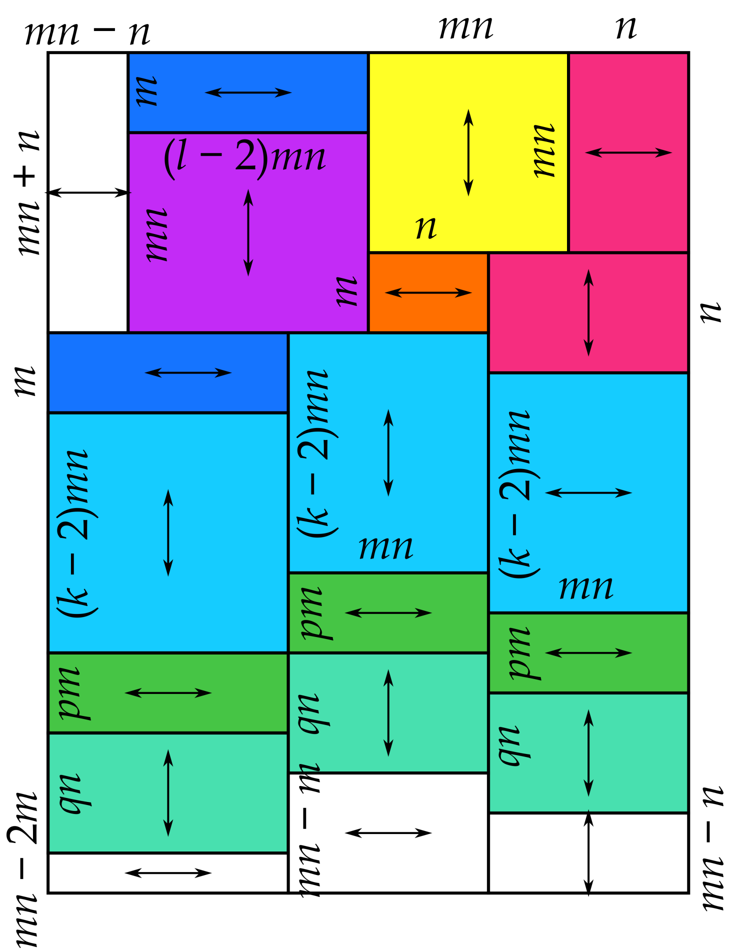


Figure 198: $kmn \times (mn + pm + qn)$, $k, l \geq 3$, $0 \leq p < n$, $0 \leq q < m$. If $q > 0$, k may be 2.

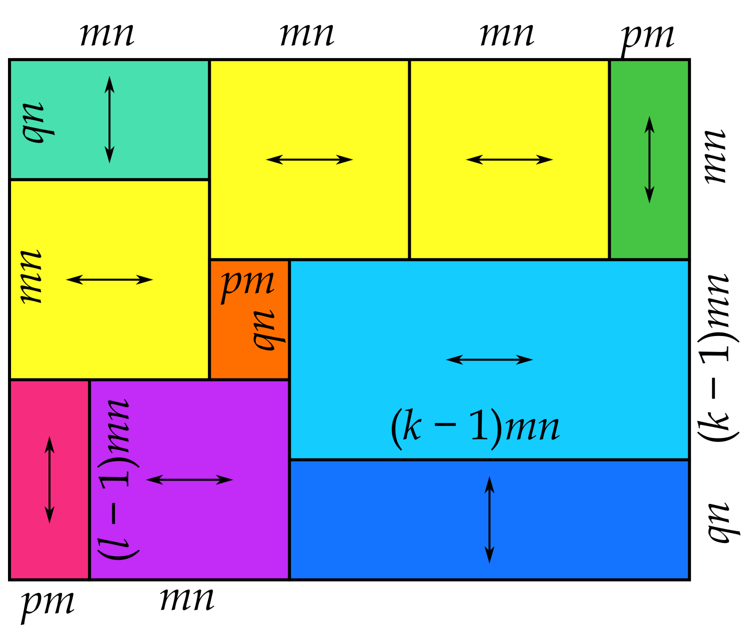


Figure 199: $(kmn + pm) \times (lmn + qn)$, with $k, l \geq 2$, $0 < p < n$, $0 < q < m$

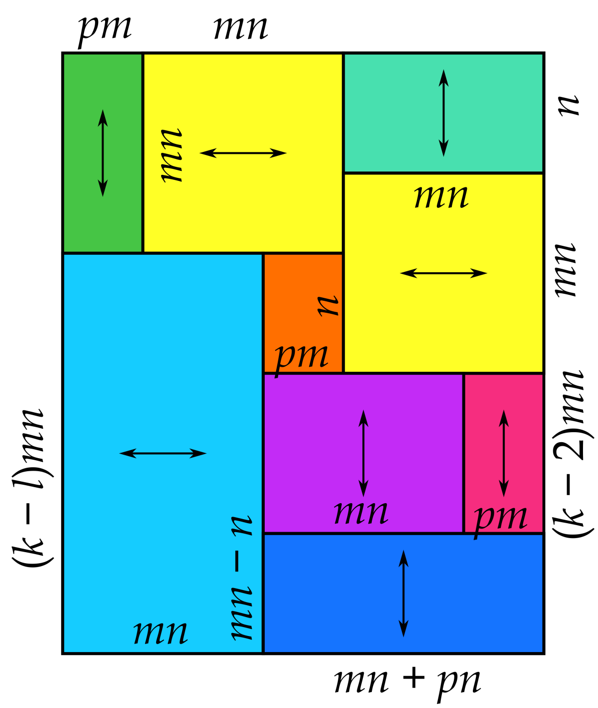


Figure 200: $kmn \times (2mn + pm)$, with $k \geq 3, 0 < p < n$

(2) $p, q \geq 2m$.

[Referenced on pages 205 and 207]

Proof. *If.* Figure 202 gives a tiling for rectangles that satisfy the conditions.

Only if. The first condition is necessary by Theorem 155.

By Theorem 224, we must have $p = x + ym$ in at least two ways. This is possibly only if $p > 2m$, in which case we can write $p = x + m = x' + 2m$. The same holds for q. Therefore, we must have $p, q > 2m$. □

Theorem 226 (Robinson (1982), Corollary 5)**.** *The minimal rectangle with a fault-free tiling is a* $3mn \times (mn + m + n)$ *rectangle*[18].

[18] This was first proven for bars in Scherer (1980)

[Not referenced]

6.4.2 Trominoes

The straight tromino is covered by Theorem 225, so we only deal with the right tromino.

Theorem 227 (Aanjaneya and Pal (2006), Theorem 2)**.** *All* $p \times q$ *rectangles with* $p, q \geq 4$, $3 \mid pq$, *admit a fault-free tiling with trominoes.*

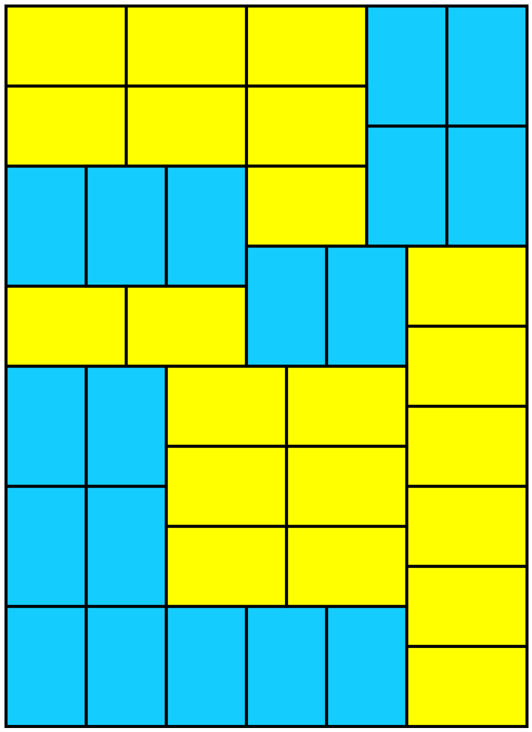


Figure 201: A fault-free tiling of 13×18 rectangle by 2×3 rectangles. The tiling was obtained from Figure fig:ff-rect1 with $k = 3$, $p = 2$, $q = 1$. Can you see how we can reduce the width of this rectangle and keep it fault-free? Why can we not reduce the height?

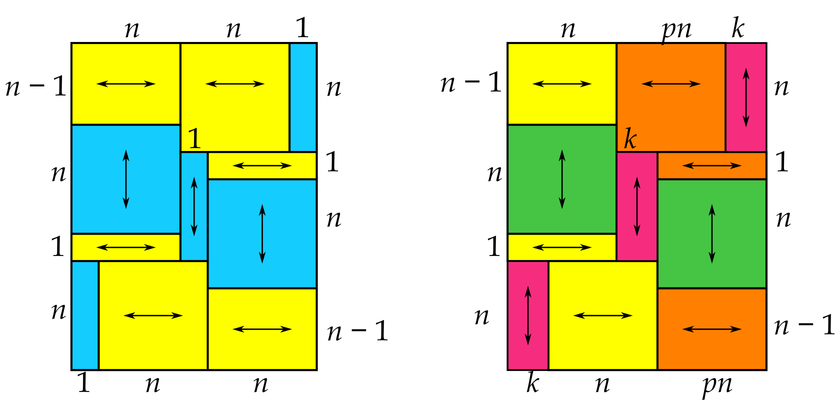


Figure 202: A fault-free tiling of $3n \times (2n+1)$ rectangle by $n \times 1$ rectangles, and an extension.

[Not referenced]

See the reference for a proof.

Since two trominoes can be put together to form a 2×3-rectangle, any rectangle that has a fault-free tiling with 2×3 tiles will also have a fault-free tiling with right trominoes. Form Theorem 224 we know this is possible for rectangles with p, q either an odd number greater than 11 or an even number greater than 14 provided at least one is divisible by three.

There are also rectangles with fault-free tilings not decomposable into 2×3 rectangles. The basic tromino fault-free tilings shown in Figure 203. These can be extended (similar to the extensions we have for dominoes) to for all other rectangles that satisfy the conditions.

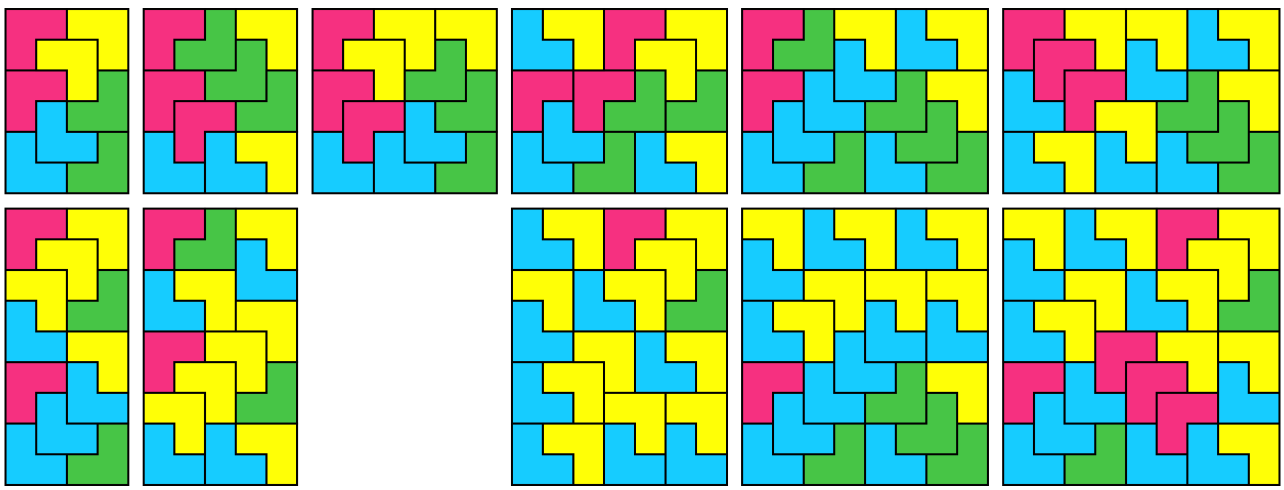


Figure 203: Basic fault free tilings using trominoes. Note that the gap is not really a gap — the missing $R(6,9)$ is present in the top-right as $R(9,6)$. Image adapted from Aanjaneya and Pal (2006, Figure 4).

All the tetrominoes except the square admit fault-free tilings. By Theorem 224 The I-tetromino has fault-free tilings for rectangles p, q with $p, q \geq 9$ and $4|(p \text{ or } q)$. The smallest can be derived from Figure 202.

6.4.3 Tetrominoes

The bar (Theorem 225) and square tetromino (Theorem 222) have already been dealt with in the previous section. The skew tetrominoe does not tile a rectangle, so here we will only consider the L and T tetrominoes.

Theorem 228. *There exist fault-free tilings by the T tetromino for any rectangle that can be tiled by the T-tetromino (that is, for all $4m \times 4n$ rectangles).*

[Not referenced]

Proof. Figure 204 shows a generic fault-free tiling. The blue sections can be repeated any number of times (including 0), leading to a ring that can be filled with 4×4 squares. This accounts for all $4m \times 4n$ rectangles, which are the only ones that the T-tetromino can tile (Theorem 208).

□

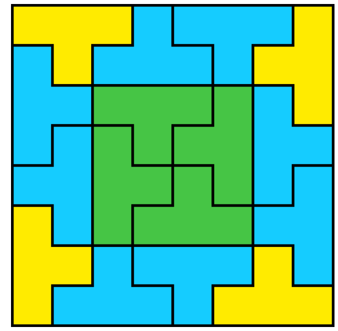


Figure 204: A generic fault-free tiling with T-tetrominoes.

Theorem 229. *Fault-free tilings by L-tetrominoes exist for all rectangles that can be tiled by L-tetrominoes, except when $m = 2$ and $n > 4$ or $n = 2$ and $m > 4$.*

[Not referenced]

Proof. Fault-free tilings exist for the following rectangles (Figure 205):

(1) 4×4

(2) 6×4

(3) 3×8

(4) 5×8

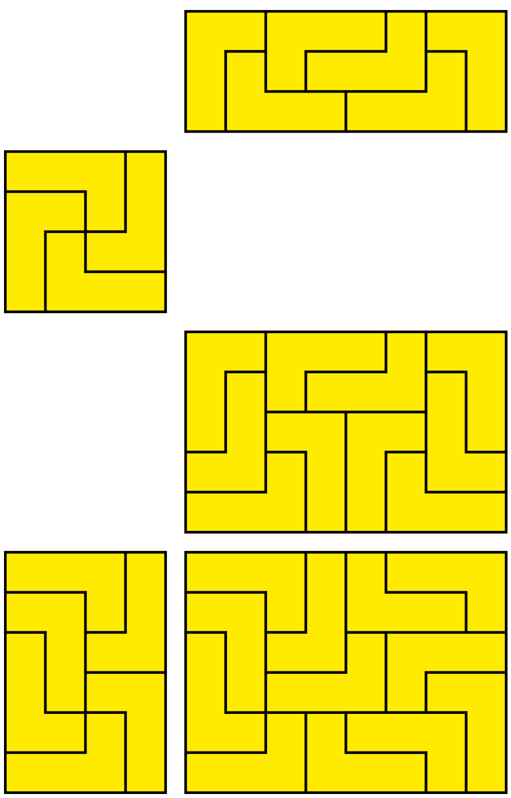


Figure 205: The basic fault-free tilings of rectangles by the L-tetromino.

These basic tilings can all be extended to give fault free tilings of the following:

(1) $4k \times 4k$

(2) $(2 + 4k) \times 4k$

(3) $(3 + 4k) \times 8k$

(4) $(5 + 4k) \times 8k$

Together, these classes account for all rectangles that can be tiled by L-tetrominoes except for rectangles with width or height 2. Of these, only the 2×4 and 4×2 are fault free.

The extensions of the first two classes can be done by inserting appropriate rectangles as shown in Figure 206.

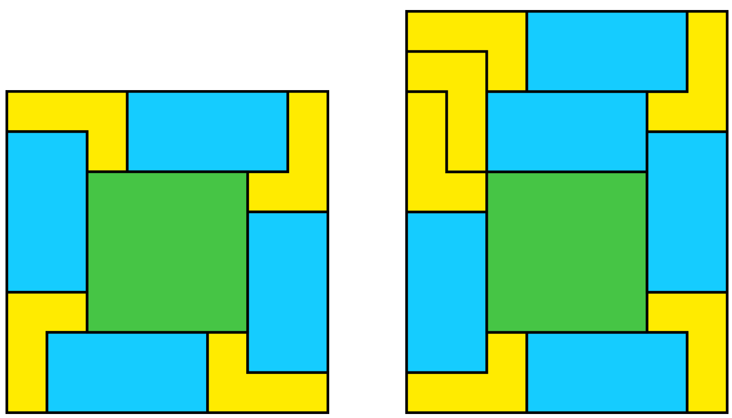


Figure 206: The standard extensions of the $4m \times 4n$ and $2 + 4m \times 4n$ rectangles.

Vertical extensions of the other two classes is done in the same way.

To extend the $3 \times 8k$ rectangle horizontally: take off one tile at the corner, and insert the shape shown in figure, and add the tile in the opening (you will need to rotate it). Doing this twice is equivalent to inserting a cylinder; we therefor call this inserting a **half cylinder**.

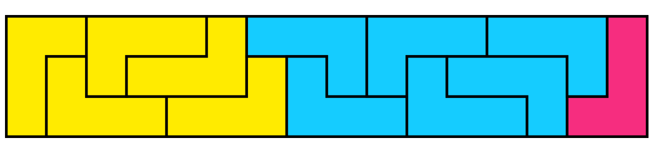


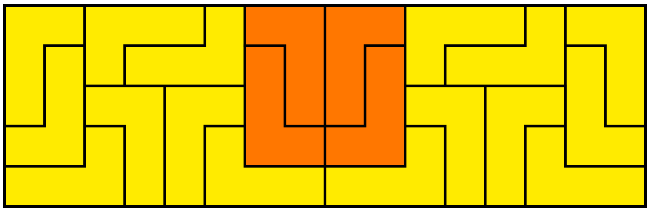
Figure 207: To extend the $3 \times 8k$ rectangle, remove the pink corner tile. Add the blue shape shown to fill the gap, and finally add the pink tile back to complete the rectangle.

To extend the $5 \times 8k$ rectangle horizontally, add a basic 5×8 rectangle, and flip the square shown in the figure.

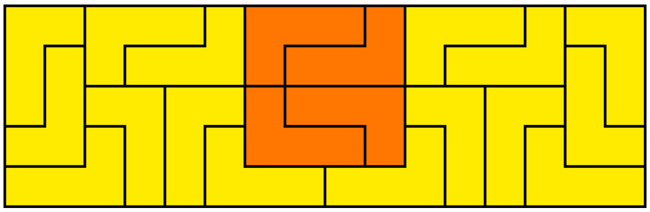
□

6.4.4 *Pentominoes*

There are only 3 pentominoes to consider: the L-pentomino, the P-pentomino, and the Y-pentomino.



(a)



(b)

Figure 208: To extend the $5 \times 8k$ rectangle, add another 5×8 rectangle, and then flip the orange square to remove the fault.

Theorem 230. *The P pentomino have fault-free tilings of all rectangles that it can tile, except for rectangles with one dimension 2 when the other is larger than 5.*

[Not referenced]

Proof. We know that one of the dimensions must be divisible by 5 (Theorem 1). It is not hard to show $5 \times m$ rectangles only have tilings if m is even.

The P-pentomino can tile the following basic rectangles fault-free:

- 2×5, 4×5, 6×5
- 10×7, 10×9, 15×7, 15×9
- 15×11, 15×11, 15×13, 15×15

See Figures 209 and 210.

These can be extended to tile the following rectangles:

(1) $2m \times 5n$ (if $m = 1$, then $n = 1$ too.)

(2) $2m + 5k \times 10n$

(3) $2m + 5 \times 5n + 5$

Extension strategies are shown in Figure 211.

This list make all rectangles with one side divisible by 5, except $5 \times m$ when m is odd, and rectangles of the form $2 \times 5n$ when $n > 1$, as is required. □

Other strategies can also be used to extend basic rectangles. Figure 212 shows an example where we put a 7×15 rectangle next to a 2×15 rectangle, and then eliminate the fault by retiling the orange tiles.

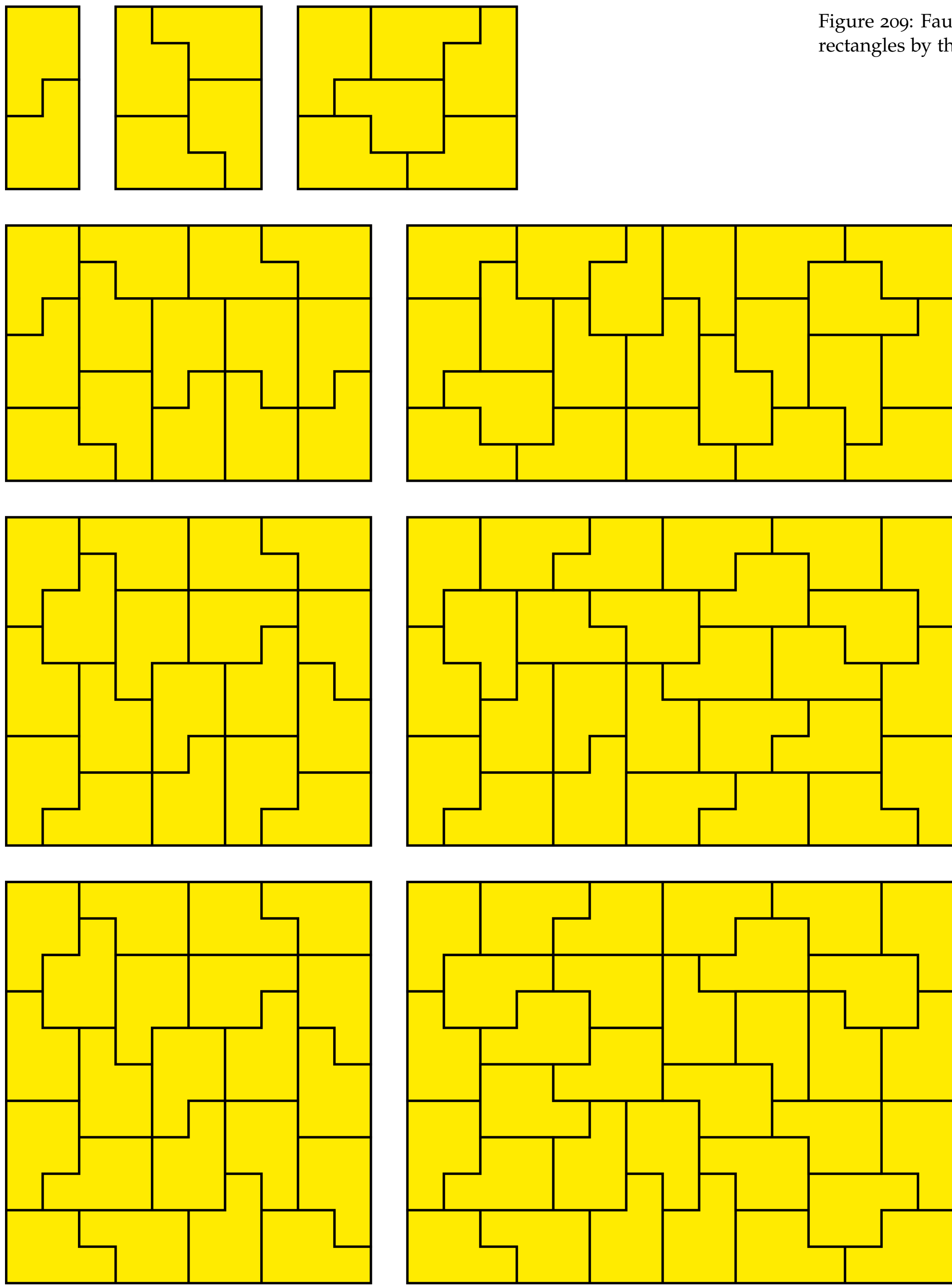


Figure 209: Fault-free tilings of basic rectangles by the P pentomino.

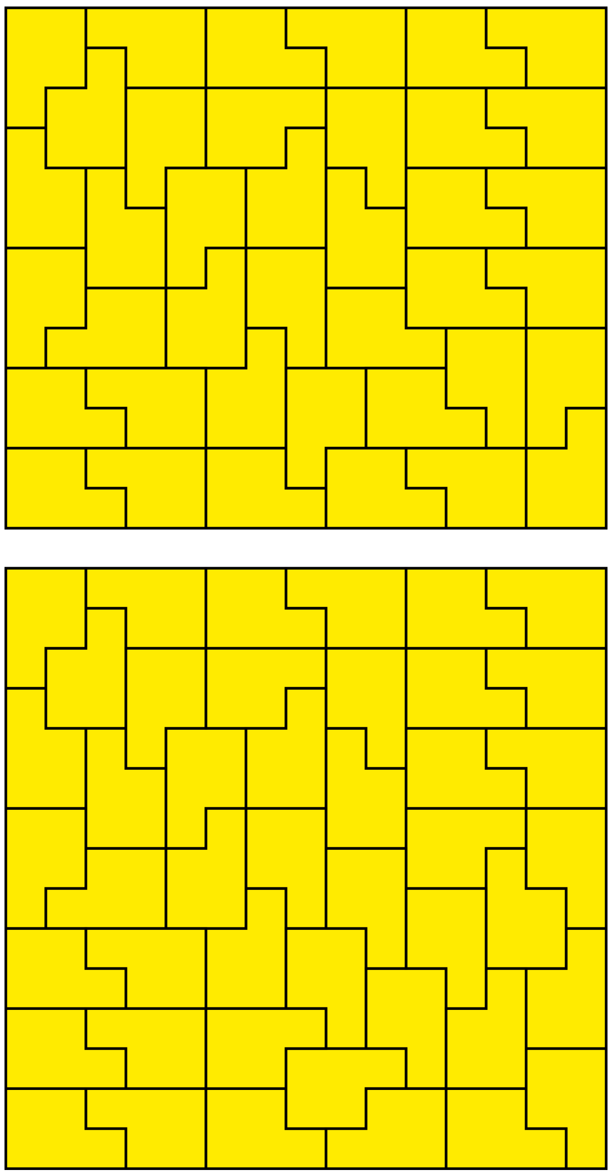


Figure 210: Fault-free tilings of basic rectangles by the P pentomino.

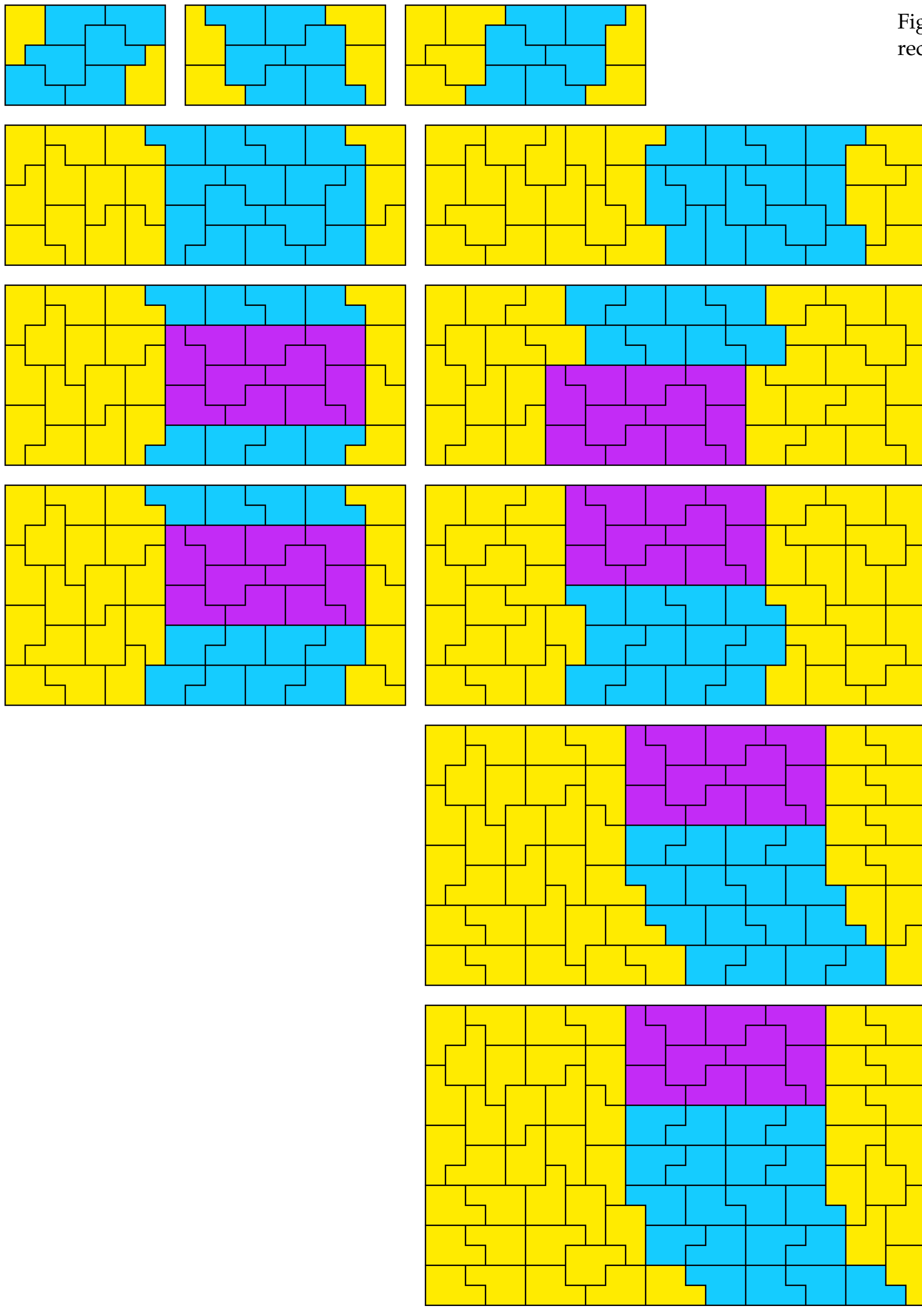


Figure 211: Fault-free tilings of basic rectangles by the P pentomino.

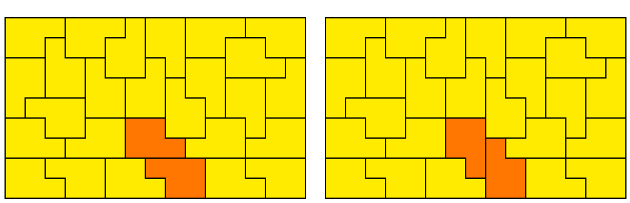


Figure 212: Fault-free tilings of basic rectangles by the P pentomino.

Theorem 231. *The L-pentomino has these basic fault-free tilings (incomplete):*

- 10×7
- 15×7
- 15×9

[Referenced on page 214]

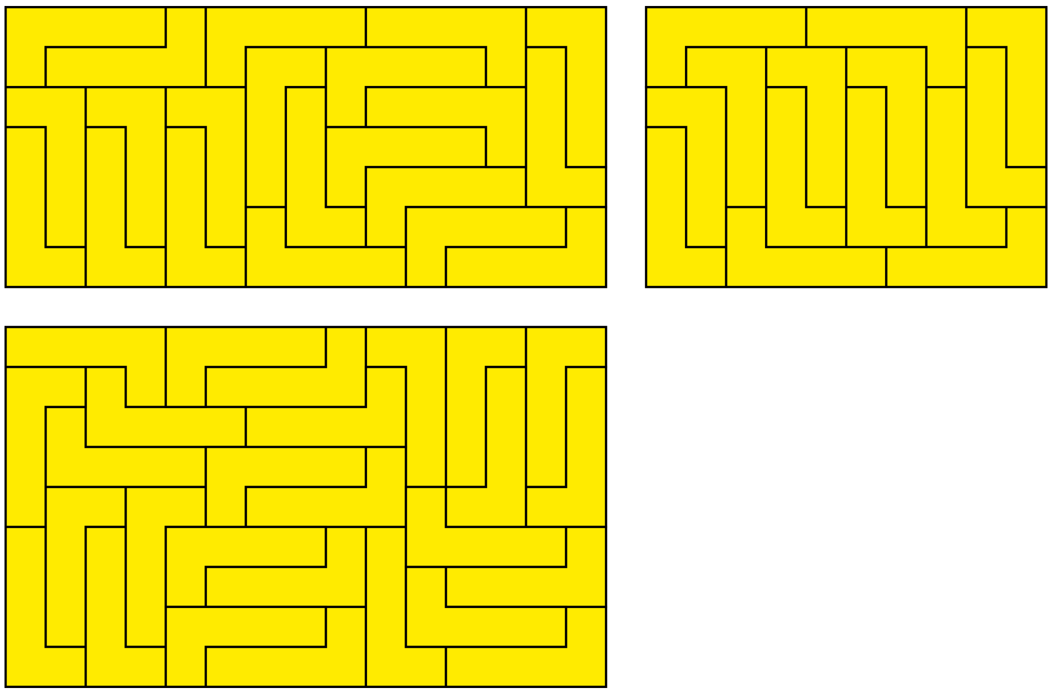


Figure 213: Basic fault-free rectangles.

Theorem 232. *The L-pentomino has these fault-free tilings (incomplete):*

(1) $10 \times (7 + 2k)$

(2) $(5k + 2p) \times 5\ell$

[Referenced on page 214]

Proof.

(1) See Figure 214.

(2) The 2×5 rectangle has a fault free-tiling, and so any fault-free tiling by $R(2, 5)$ will also be fault-free if replace the rectangles with the tilings by the pentomino. The fault-free tilings of $R(2, 5)$ is given by Theorem 223.

□

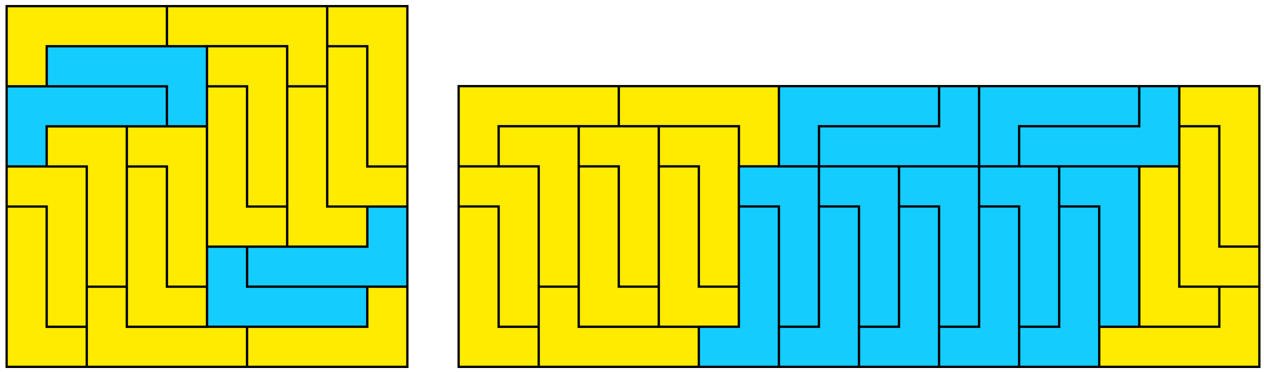


Figure 214: Fault-free extensions.

Problem 57. *Complete the lists of Theorems 231 and 232.*

Finding all the rectangles with fault-free by Y-pentominoes is more tricky. Some extensions is shown in Figures: 215–222. Extension like these are also given in Cibulis and Mizniks (1998).

Problem 58. *Find all the fault-free tilings of rectangles by the Y-pentomino.*

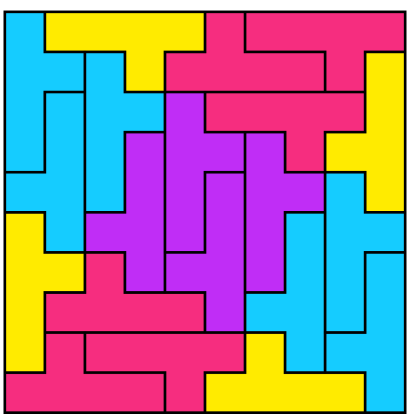


Figure 215: Example of an extension of a fault-free rectangle. We can duplicate the blue-purple region n times (including 0) to extend the rectangle vertically; as shown in the next three figures. We can also repeat the pink-purple region to extend the rectangle horizontally.

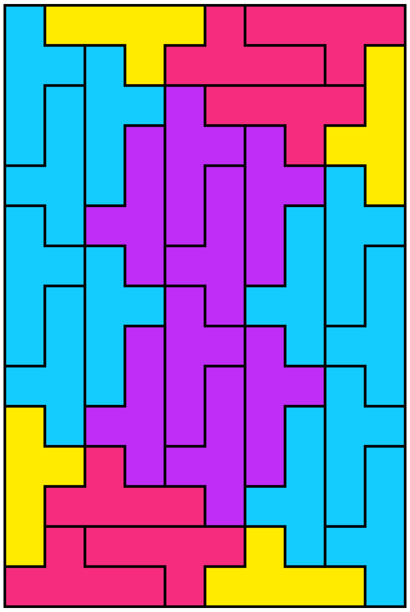


Figure 216: Example of an extension of a fault-free rectangle.

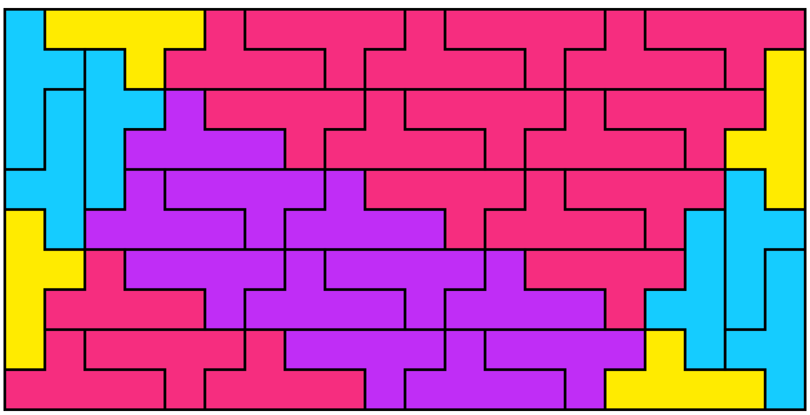


Figure 217: Example of an extension of a fault-free rectangle.

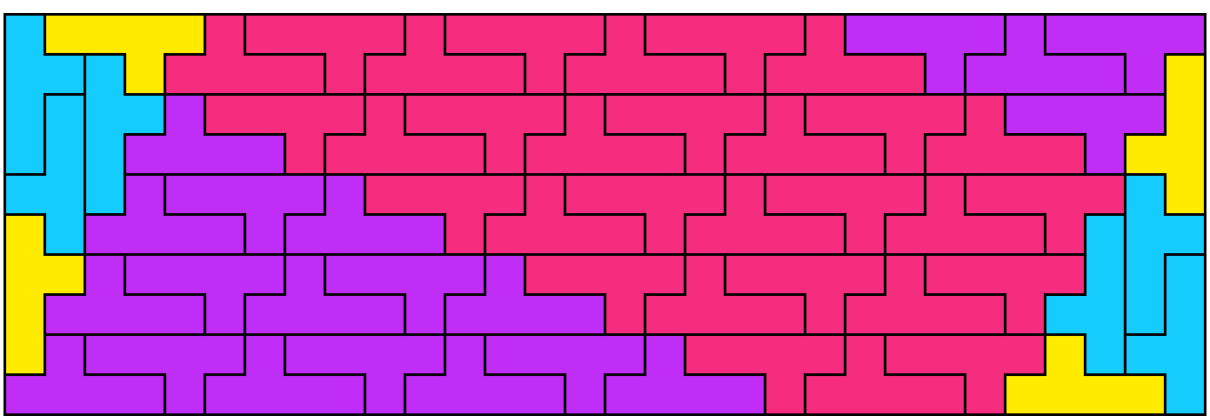


Figure 218: Example of an extension of a fault-free rectangle.

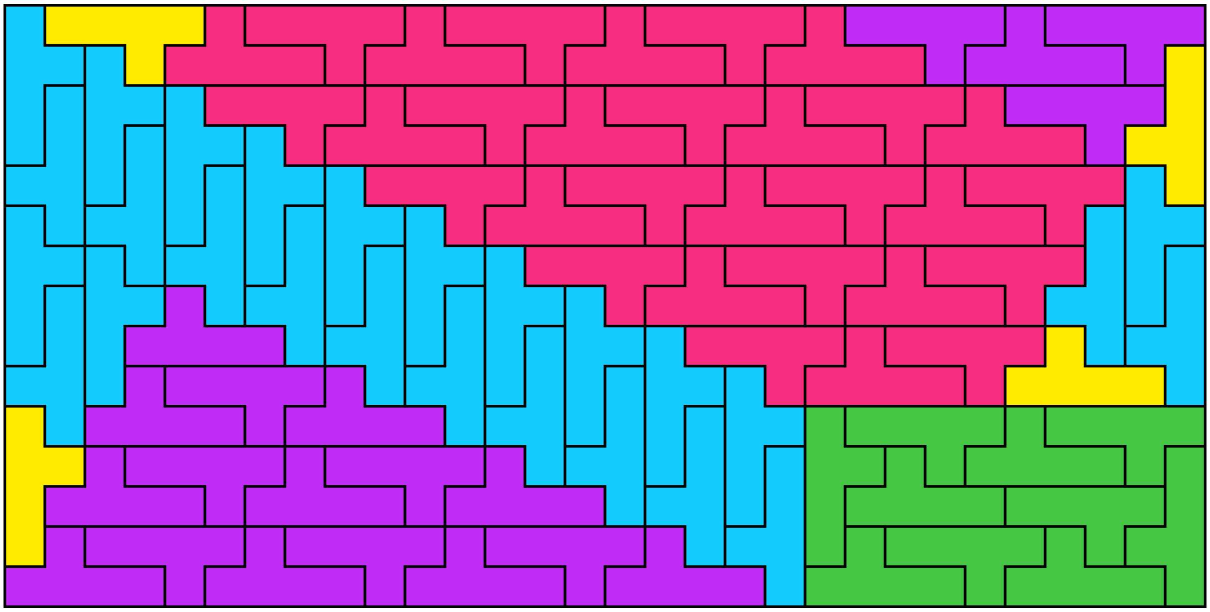


Figure 219: Example of an extension of a fault-free rectangle.

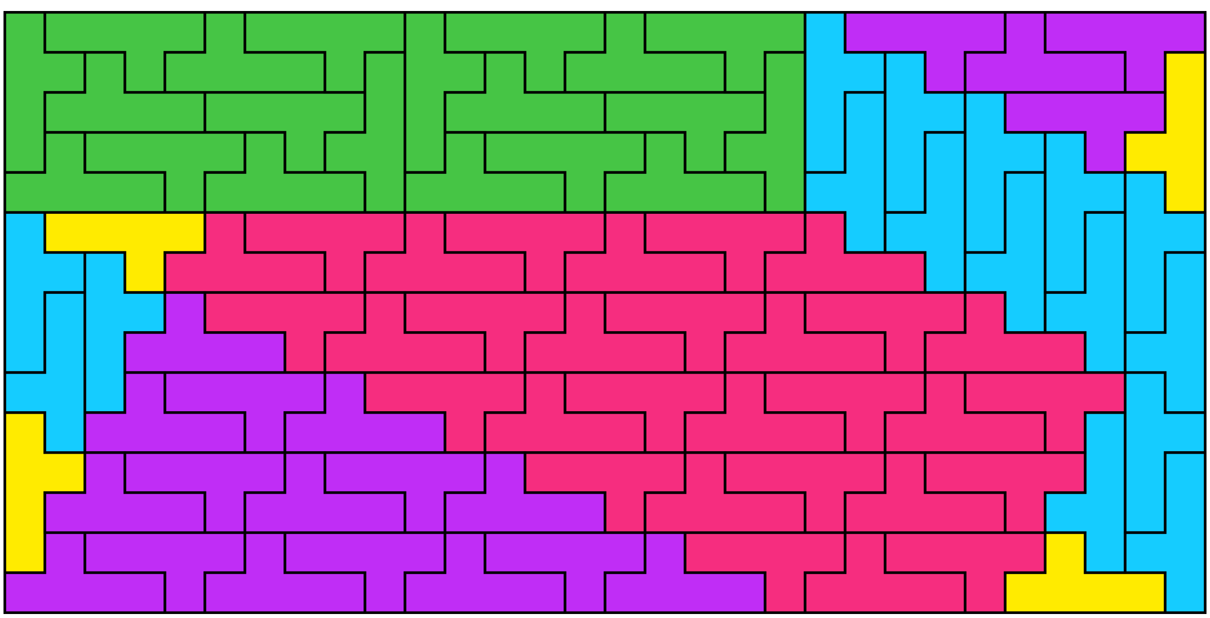


Figure 220: Example of an extension of a fault-free rectangle.

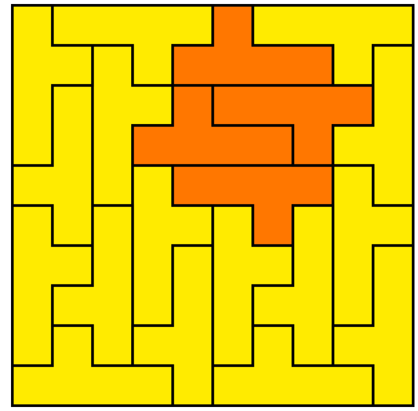


Figure 221: Another way to extend the 5×10 rectangle. Place two copies next to each other, and rotate the orange tiles as shown to eliminate the fault.

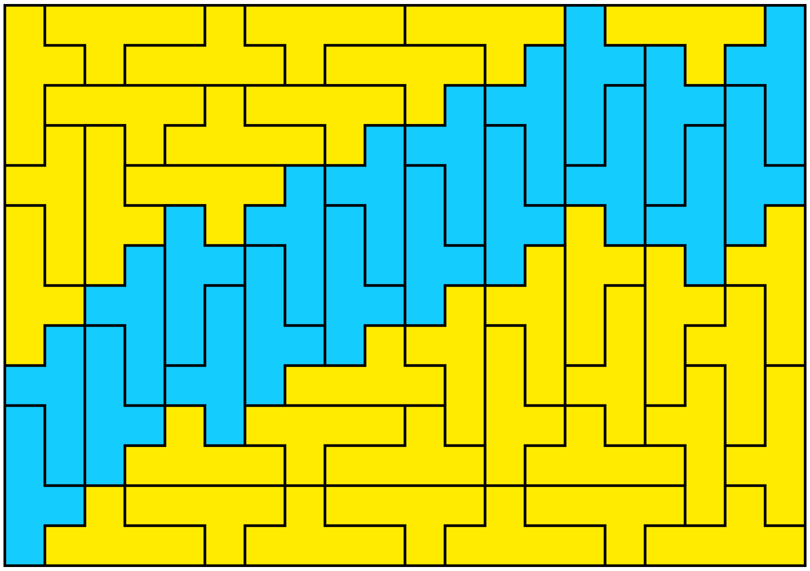


Figure 222: Example of an extension of a fault-free rectangle.

6.4.5 *Extensions*

We saw a few ways that rectangles can be extended:

- Through cylinder insertion.
- Through half-cylinder insertion.
- Through concatenation combined with fault-elimination (by retiling a subregion).

6.5 *Simple Tilings*

A tiling of a figure with more than two tiles is **simple** if no subset of tiles (with more than one tile) form a rectangle strictly inside the figure. For rectangles, simple tilings are also fault free (otherwise, the rectangle will split into two rectangles at the fault, at least one of which must have more than one tile).

Theorem 233 (Martin (1991), p. 15)**.** *No rectangle has a simple tiling with dominoes.*

[Not referenced]

Proof. It is obvious for $R(m, 1)$ or $R(1, n)$.

A rectangles $R(m, n)$ with $m, n > 1$ have no peaks or holes, and therefor any tiling of it has a flippable pair (which is a rectangle) as subtiling (Theorem 65), and is therefore not simple. □

Theorem 234 (Martin (1991), p. 15)**.** *Any simple tiling of the plane with dominoes is fault-free.*

[Not referenced]

Proof. Suppose there is a fault, WLG let it be a horizontal fault. The pattern on the fault must be alternating vertical and horizontal dominoes, since any two adjacent dominoes of the same orientation will form a rectangle (and therefor the tiling would not be simple.) Therefor, we can find an LR-configuration, which means there is a forced flippable pair inside the induced pyramid (Theorem 96), which means the tiling cannot be simple. Therefor, there can be no fault-line. □

Theorem 235 (Martin (1991), p. 16)**.** *The plane has a unique (up to reflection) simple tiling with dominoes.*

[Not referenced]

Proof. WLG, suppose the tiling contains a vertical domino, with 6 neighbors, labeled 1–6, starting from the bottom, going anticlockwise.

(1) In position 1, we cannot fit a vertical domino, so it must fit a horizontal domino. Let's say it lies to the right.

(2) Similarly, in position 4, we must fit a horizontal domino. It cannot also lie to the right, since otherwise we will have an LR configuration, which will force a flippable pair (Theorem 96), therefor it must lie to the left.

(3) In position 2, we cannot fit a vertical domino (since it will either overlap with the domino on 1, or form a rectangle), therefor it must be horizontal.

(4) A similar argument applies to position 5.

(5) In position 3, a horizontal domino will form a rectangle with the domino at position 2, therefor it must be vertical.

(6) A similar argument applies to position 6.

We can now repeat the same argument for the new vertical dominoes, except that because we have some neighbors placed, we do not have a choice we had in step (1). And a symmetrical procedure can be applied to horizontal dominoes; the entire tiling is forced.

If we made a different choice in step (1) above, we will construct the same tiling, but reflected.

□

If we now consider general polyominoes, we can find simple tilings with 5 (Figure 231) and 7 (Figure 232) tiles, and as the following theorem shows, for all numbers of tiles 7 or bigger.

Theorem 236 (Los et al. (1960)). *There exists a simple tiling with n tiles for all $n \geq 7$.*

[Not referenced]

Proof. Consider the construction shown in Figure 223. The rectangles are placed in a spiral as shown. To get a simple tiling with n tiles, place the first n tiles in the spiral as shown, but truncate the last tile so that it does not extend beyond the rectangle. This works for all $n \geq 5, n \neq 6$. (It does not work for $n = 6$, because truncating the sixth tile will make it the same size and aligned with the second tile.) □

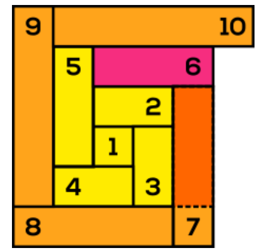


Figure 223: A construction showing how we can get a simple tiling for all $n \geq 5, n \neq 6$. For example, to get a simple tiling with 7 tiles, place the first 7 tiles in the spiral as shown, and truncate it at the dotted line.

Proof. (Alternative proof (Chung et al., 1982, Theorem 1).) Using the blocks in Figure 224 we can construct simple tilings using any number of tiles greater than 7 by choosing an appropriate left and right block, and any number (including 0) of center blocks. An example construction is shown in Figure 225. □

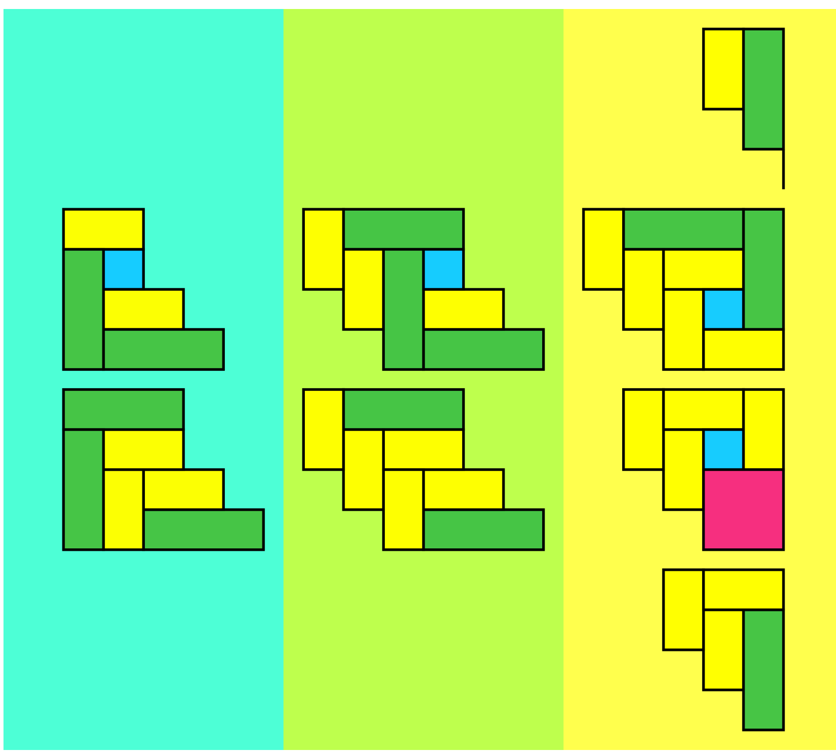


Figure 224: Blocks that can be used to construct simple tilings using n pieces for all $n \geq 7$.

Theorem 237. *There is no simple tiling with 6 rectangles.*

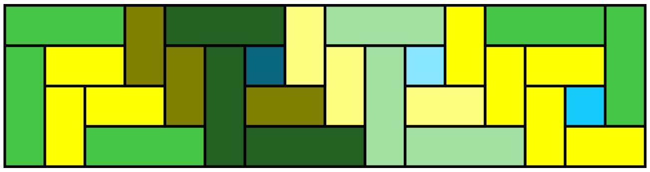


Figure 225: An example construction of a simple tiling with 28 tiles.

[Not referenced]

Proof. (Adapted from Taxel (2016).)

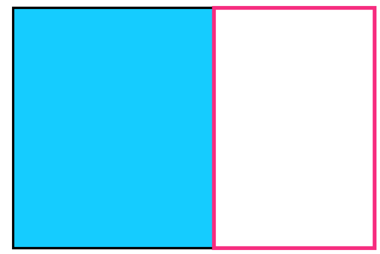
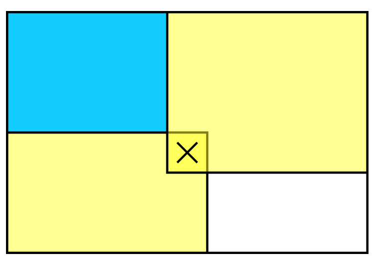
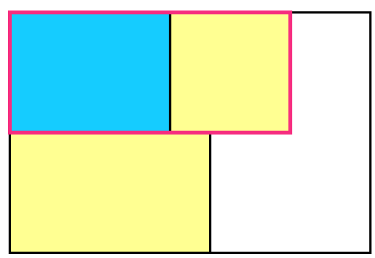


Figure 226:

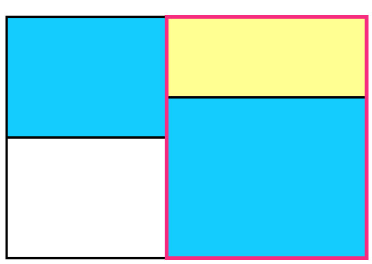
(1) In a simple tiling, no rectangle can cover two corners of the completed figure. If there were such a piece, the remaining pieces would form a rectangle. Therefor, the four corners must be covered by different rectangles.

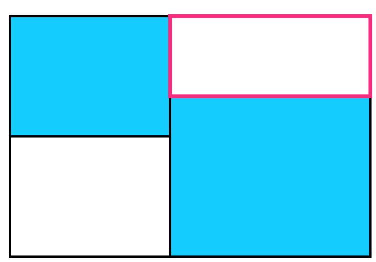
(2) A corner piece must have at least 3 neighboring pieces. If a corner piece had two neighbors, they have to lie on opposite sides. If both neighbors were longer than the side, they must overlap. Therefor, at least one is the same size. But then it makes a rectangle with the corner piece, which is not possible since the tiling is simple.





(3) Diagonally opposite corner pieces cannot touch. If they did, the remaining figure would be two rectangles. If either rectangle is a single piece, it forms a rectangle with one of the corner pieces. Otherwise, it is tiled by smaller pieces, which violates the tiling being simple.





Now lets consider $n = 6$ specifically. Four of those six pieces must be in the four corners of the final figure (1).

Suppose the remaining two pieces are fully internal to the figure. Each of the outside pieces can only expose one side to the internal area, so the internal area is rectangular. Filling it with the remaining two pieces creates a sub-rectangle with 2 pieces.

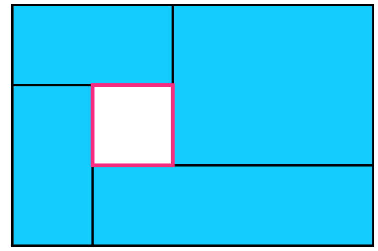


Figure 227:

Suppose on the other hand that all 6 pieces are on the boundary of the final figure, i.e. there are no internal pieces. So we have 4 corner pieces and 2 edge pieces.

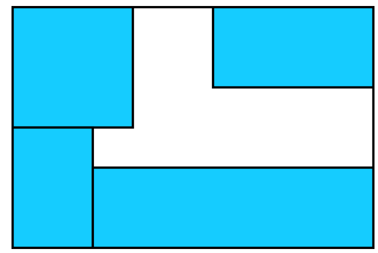
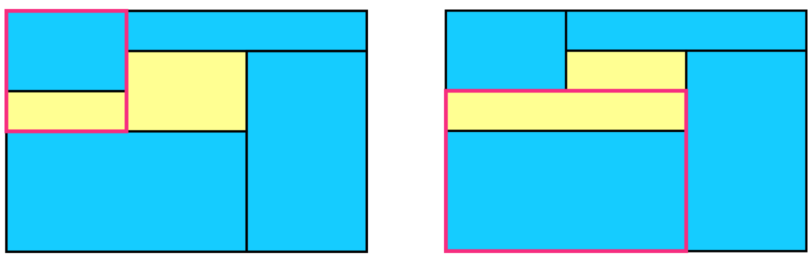


Figure 228:

- Suppose a corner piece lies between two edge pieces. It must have a third neighbor (2), but the only candidate is the diagonally opposite corner, which violates (3).

- Suppose the two edge pieces are adjacent. The four corner pieces must be arranged so that the opening is in L-shape. There are two ways to fill the L-shape with rectangles; in each case one rectangle must be interior so both cannot be edge pieces and therefor this arrangement is impossible.[19]

[19] The original proof omits this case.



- The only other arrangement for the edge pieces is on opposite sides of the final figure, say the left and right sides. The two top corners are adjacent, cannot touch either of the bottom corners, so the only way for them to have 3 neighbors is for both corners to be adjacent to both edge pieces. This is not possible.

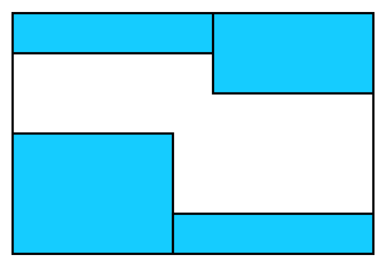


Figure 229:

The last possibility is that we have 4 corner pieces, 1 edge piece, and 1 internal piece. The two corners next to the edge piece must have the internal piece as their third neighbor. The edge piece has three internal sides and so must have at least three neighbors. The only possibility is that it is also adjacent to the internal piece. In a similar argument to (2), the corners cannot be the same length as the edge piece, and if both were longer then the edge piece and the internal piece have matching lengths and form a rectangle.

All possibilities lead to failure, so $n = 6$ is impossible. □

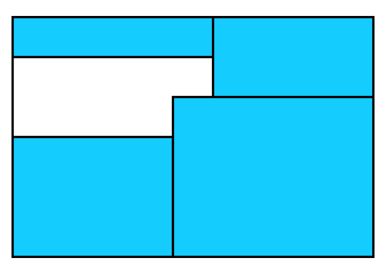


Figure 230:

The **average area** of a tile in a tiling of a region R is given by $|R|/n$, where R is scaled so that the smallest tile has area 1, and n is the number of tiles[20].

[20] We always stretch the rectangle so that the smallest tile is a square.

Theorem 238 (Chung et al. (1982), Theorem 2)**.**

(1) *Except for the simple tiling in Figure 231 that has average area* $9/5$*, all other simple tilings of rectangles has average area stricter greater than* $11/6$*.*

(2) *There is a unique simple tiling of the plane with average area of* $11/6$*. This tiling is shown in Figure 234.*

[Not referenced]

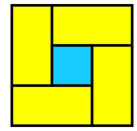


Figure 231: A simple tiling with 5 tiles.

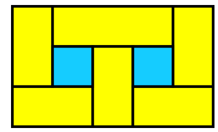


Figure 232: A simple tiling with 7 tiles.

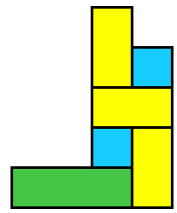


Figure 233: A tile that can be used to construct the tiling shown in Figure fig:rect-simple-plane

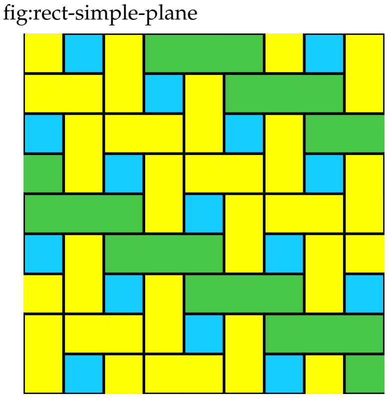


Figure 234: A tiling of the plane with average area is $11/6$.

6.6 Further Reading

Theorem 159 is given in a slightly more general form in the original paper (where it applies to n-dimensional bricks with arbitrary lengths, not necessarily integers). Wagon (1987) gives and discuss 14 proofs of this theorem, including some generalizations.

Some problems relating to tiling rectangles by rectangles:

(1) *Finding square tilings of squares.* When each square has a different area, the tiling is called *perfect*. Anderson (2013) gives a chronology of squaring the square discoveries and is a good introduction to the problem.

(2) *Finding whether a set of rectangles will pack a given rectangle.* A survey of this is given by Lodi et al. (2002).

The question to show that there exists an integer C such that $R(m,n)$ is tileable by $R(4,6)$, $R(6,4)$, $R(5,7)$ and $R(7,5)$ for all $m,n > C$ was asked in the *The Fifty-Second William Lowell Putman mathematical competition* (Klosinski et al., 1992, Problem B-3). Narayan and Schwenk (2002) finds the minimum value C, and the thesis of Dietert (2010) considers which rectangles can be tiled by this set.

Barnes (1982a) and Barnes (1982b) considers tilings of bricks by bricks from a algebraic viewpoint. The n-dimensional version of Theorem 171—the gap number bricks packed by rods —is proven in Barnes (1995). In Brualdi and Foregger (1974) the problem is discussed for harmonic bricks, and in Barnes (1979) for sufficiently large" rectangles .

Frobenius numbers and related problems are covered in detail in Ramírez-Alfonsín (2005).

Theorem 201 has been generalized by Maltby (1994) by showing that trisecting a rectangle into 3 congruent pieces of any shape (not necessarily polyominoes), implies the pieces are rectangles. Yuan et al. (2016) showed that if a square is dissected into 5 convex congruent pieces, there is a unique solution with rectangles only. (Unfortunately, this is not helpful for polyominoes, since the only polyominoes that are convex in the normal geometric sense are rectangles.)

Kenyon (1996) gives an algorithm for determining whether a rectilinear figure (not necessarily with all integer sides) is tileable with rectangles each of which has at least one integer side. One drawback of this algorithm is that the tiling rectangles cannot be specified as part of the input.

For more on tilings by bars, see Beauquier et al. (1995).

Bloch (1979) contains tilings by rectangles for small rectangles. Tables with the number of tilings for small polyominoes can be found in Grekov (http://polyominoes.org/data).

For more on prime rectangles see de Bruijn and Klarner (1975), Klarner (1973), and Reid (2005). The latter also gives many examples of finding prime rectangle for polyominoes and polyomino sets. Reid (2008) discusses tiling theorems for "sufficiently large" rectangles.

Korn and Pak (2004) consider T-tetromino chains and height functions of T-tetromino tilings. The number of tilings of rectangles by T-tetrominoes is considered in Merino (2008). Tilings of deficient rectangles by L-tetrominoes are discussed in Nitica (2004), and by T-tetrominoes in Zhan (2012). Gap numbers for the T-tetromino on rectangles is covered in Hochberg (2015).

7 Plane Tilings

When and how polyominoes tile the plane are the central questions of this section.

In many ways, plane tilings are more fundamental than tilings of rectangles; for one thing — any tile set that tiles a rectangle can also tile the plane.

Theorem 239 (Grünbaum and Shephard (1987), Chapter 3)**.** *In a tiling by any tile set $\mathcal{T}$, there must be a tile with at most 6 neighbors.*

[Referenced on page 228]

A tiling is called **monohedral** if all the tiles in the tilesets are congruent[1] (Grünbaum and Shephard, 1987, p. 20).

[1] The terminology makes sense when applied to tilings of other regions, but is generally only used to discuss plane tilings.

Suppose there are two non-parallel non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $U + \boldsymbol{u} + \boldsymbol{v}$ is the same tiling, then the tiling is periodic. If there is only one such vector, the tiling is called half-periodic. If there is no such vector, the tiling is **non-periodic**.

Example 19. *We can easily form half-periodic and non-periodic tilings of the plane from tiles using irrational numbers. Here are some examples using dominoes:*

- *Form a half-strip using vertical and horizontal flippable pairs, using alternate digits of π of each; that is, 3 vertical, 1 horizontal, 4 vertical, 1 horizontal, etc. Reflect the strip and form a complete strip; now stack these together to tile the plane. The resulting tiling is half-periodic.*

- *Form a strip using the same scheme as above; form another strip reversing the roles of vertical and horizontal flips. Now stack these together using alternate digits of π for each.*

Another method is to use a tiling with all dominoes vertical (and aligned). Flip a single flippable pair; this tiling is non-periodic. If we flip all the dominoes in a row, we get a half-periodic tiling.

Some tile-sets can only tile the plane non-periodically. Such a tile-set is called **aperiodic**. If the set is a single tile, we call the tile aperiodic. Currently, we don't know if connected aperiodic tiles (polyominoes or non-polyominoes) exist.[2]

[2] See Socolar and Taylor (2011) for an example of a disconnected aperiodic tile.

Informally, a **fundamental region** of a tiling is the smallest part of the tiling that we can translate to form the entire tiling. More formally, the fundamental region of a periodic tiling with period $\boldsymbol{u}, \boldsymbol{v}$ is a set F of connected cells such that if $x \in F$, then $x + m\boldsymbol{u} + n\boldsymbol{v} \neq F$ for all $m \neq 0, n \neq 0$. [3]

[3] The fundamental region is also called a *fundamental domain*. Note, this is different from the definition given in for example Grünbaum and Shephard (1987, p. 55), and (Kaplan, 2009, Definition 3.1, p. 17) that is defined in terms of symmetry groups.

7.1 Tilings by Translation

A polyomino that can tile the plane by translation alone is called **exact** (Beauquier and Nivat, 1990, Section 1). The smallest tiles that are not exact are the pentominoes: F, U and T.

Theorem 240 (Translation Criterion, (Rhoads, 2005), Theorem 2)**.** *A tile admits a tiling of the plane if its border can be divided into six segments, A, B, C, D, E and F such that the pairs $A - D$, $B - E$ and $C - F$ are translates of each other. (Both edges in one of these pairs may be empty).*

[Not referenced]

If word that can be written in the form $ABC\hat{A}\hat{B}\hat{C}$ (possibly with one pair of factors empty), we call that form the BN-factorization of the word. A word can have more than one BN factorization.

Theorem 241 (Beauquier-Nivat Criterion, Winslow (2015))**.** *A polyomino is exact if its border has a BN factorization . .*

[Referenced on page 226]

7.2 Tilings by Translation and Rotation

Problem 59. *Show that no The only Young diagrams that are exact are rectangles and L-shaped polyominoes.*

Problem 60.

(1) *Find the other factorizations of X_5.*

(2) *Suppose P is a polyomino with border word $x^{\pm 1}y^{\pm 1}x^{\pm 1}y^{\pm 1} \cdot x^{\pm 1}y^{\pm 1}$. Characterize the ones with a BN factorization..*

Theorem 242. *The families of polyominoes listed in table 36 are all exact.*

[Not referenced]

Polyomino		BN Factorization
X_5		$(xyx)(yx^{-1}y)\cdot(x^{-1}y^{-1}x^{-1})(y^{-1}xy^{-1})$
Z_5		$(x)(y)(x^{-1}y^2x^{-1})\cdot(x^{-1})(y^{-1})(xy^{-2}x)$
N_5		$(xy^{-1}x^2)(x)(y)\cdot(x^{-2}yx^{-1})(x^{-1})(y^{-1})$
$R(m,n)$	Rectangle	$(x^m)(y^n)\cdot(x^{-m})(y^{-n})$
$A(n)$	Aztec diamond	$(xy)^n(yx^{-1})^n\cdot(x^{-1}y^{-1})^n(y^{-1}x)^n$
$B(a^m\cdot b^n)$	L-shaped polyomino	$(y^{n-m}x^b)x^ay^n\cdot(x^{-b}y^{m-n})x^{-a}y^{-n}$
W_{2k}	Even-area W-polyomino	$x\left[(xy)^{k-1}x\right]y\cdot x^{-1}\left[(x^{-1}y^{-1})^{k-1}x^{-1}\right]y^{-1}$
W_{2k+1}	Odd-area W-polyomino	$x(xy)^ky\cdot x^{-1}(y^{-1}x^{-1})^ky^{-1}$
Y_n	General Y-polyomino	$x(yx^3)(yx^{-1})\cdot x^{-1}(x^{-3}y^{-1})(xy^{-1})$
$B(1^k\cdot 2^m\cdot 1^n)$		$x^m(yx^n)(yx^{-k})\cdot x^{-m}(x^{-n}y^{-1})(x^ky^{-1})$
$C_k(a_1,a_2,\cdots,\cdot a_m)$	Cylinder	$x^kA\cdot x^{-k}\hat{A}$, where $A=x^ky^{a_1}x^ky^{a_2}\cdots x^ky^{a_m}$

Table 36: BN factorizations for various polyominoes and families of polyominoes.

Proof. The BN factorization is given in the table for each family, which makes it exact (Theorem 241). □

Note that this proves exactness for all n-ominoes with $n\leq 5$, except for F,

Theorem 243 (Conway's Criterion)**.** *A polygon admits a periodic tiling of the plane using only translations and* $180°$ *rotations if it has 6 points (at least three of which are distinct), that satisfy these conditions:*

(1) *The boundary between A and B is congruent by translation to the boundary between E and D.*

(2) *The boundaries BC, CD, EF and FA are all centrosymmetric.*

[Not referenced]

Problem 61. *Show that the golygon with 8 sides (Figure 14) satisfies the Conway criterion.*

7.3 Isohedral and k-Isohedral Tilings

For two congruent tiles in a tiling there must be at least one rigid motion of the plane that maps the one tile to the other. If this motion is also a symmetry of the plane, then this motion maps the whole tiling to itself, and the two tiles are **transitively equivalent**.

Transitive equivalence is a equivalence relation, that divides the tiles in the tiling into **transitivity classes**. If there are k classes, we call the tiling k**-isohedral**[4]; if $k=1$, we simply call the tiling **isohedral**[5].

[4] Also *tile-k-transitive* (Huson, 1993, Section1, p.272).

[5] Also *tile-transitive* (Huson, 1993, Section1, p.272).

There are 93 kinds of isohedral tilings of the plane with marked tiles, distinguished by how a tile relates to its neighbors. 81 of these can be realized with tiles without markings. (Grünbaum and Shephard (1977), Grünbaum and Shephard (1987, 6.2.1)) 67 of the 93 tilings can be realized with polyominoes; 60 can be realized with polyominoes without markings. The 81 kinds of isohedral tilings can all be described by 9 rules given in Table 37.[6]

[6] According to Langerman and Winslow (2015, Section 3) these are first shown in Heesch and Kienzle (2013, Table 10), and reproduced in Schattschneider and Escher (1990, p. 326).

Theorem 244. *Each tile in an isohedral tiling has at most 6 neighbors.*

[Not referenced]

Proof. By Theorem 239 there must be at least one tile with at most 6 neighbors. But since transformations that map any other tile to this one maps the tiling to itself, it means all tiles must have at most 6 neighbors. □

In general, a single tile can belong to different transitivity classes in a single tiling. Some examples are shown. Therefor, being k-isohedral is a property of the tiling and not the tile set. However, a tile set that admits k-isohedral tilings, but no m-isohedral tilings for $m < k$, is called **k-anisohedral**.

To summarize: tilings can be k-isohedral; tile sets can be k-anisohedral.

There is k-anisohedral polyominoes for all k between 1 and 6 (shown in Figures 235-240). We do not know whether polyominoes exist for other k (Winslow, 2018, Problem 6).[7]

[7] We do know that other tiles exist for $k = 8, 9, 10$. See for example Myers.

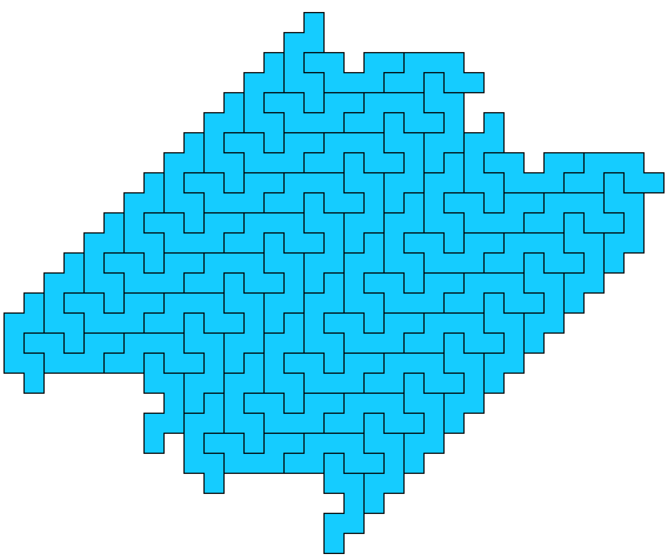


Figure 235: An isohedral tile that does not satisfy Conway's Criterion. (Myers)

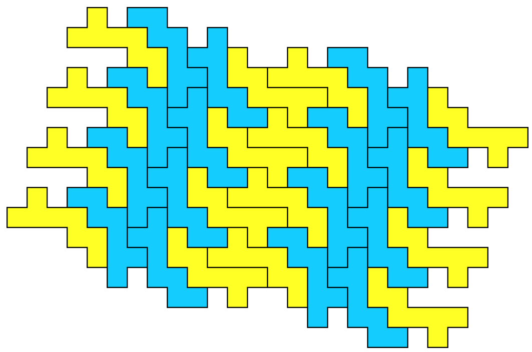


Figure 236: An 2-anisohedral tile. Transitivity classes are shown in the same color. (Myers)

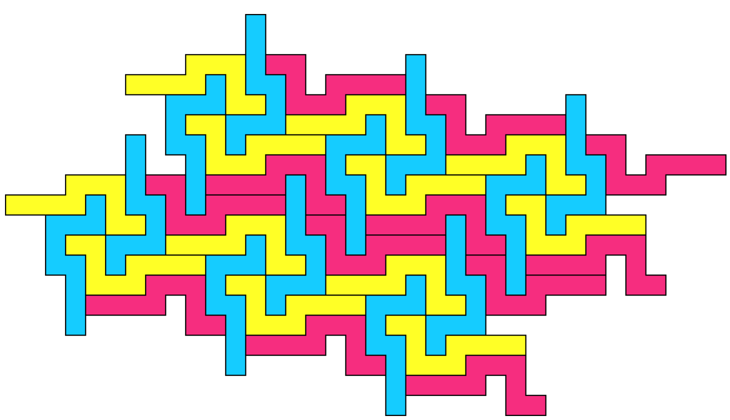


Figure 237: An 3-anisohedral tile. Transitivity classes are shown in the same color. (Myers)

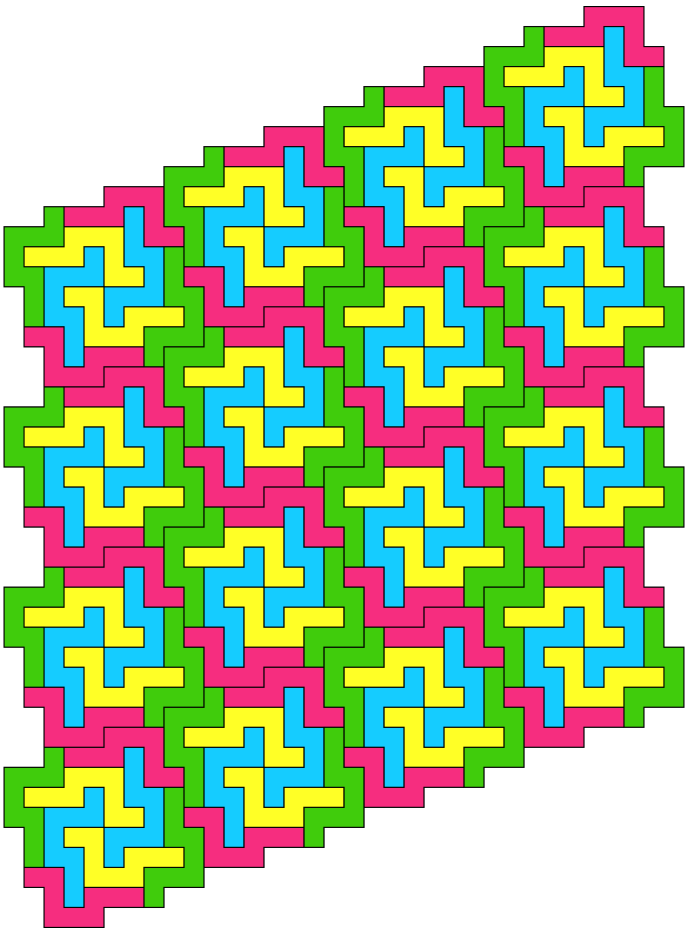


Figure 238: An 4-anisohedral tile. Transitivity classes are shown in the same color. (Myers)

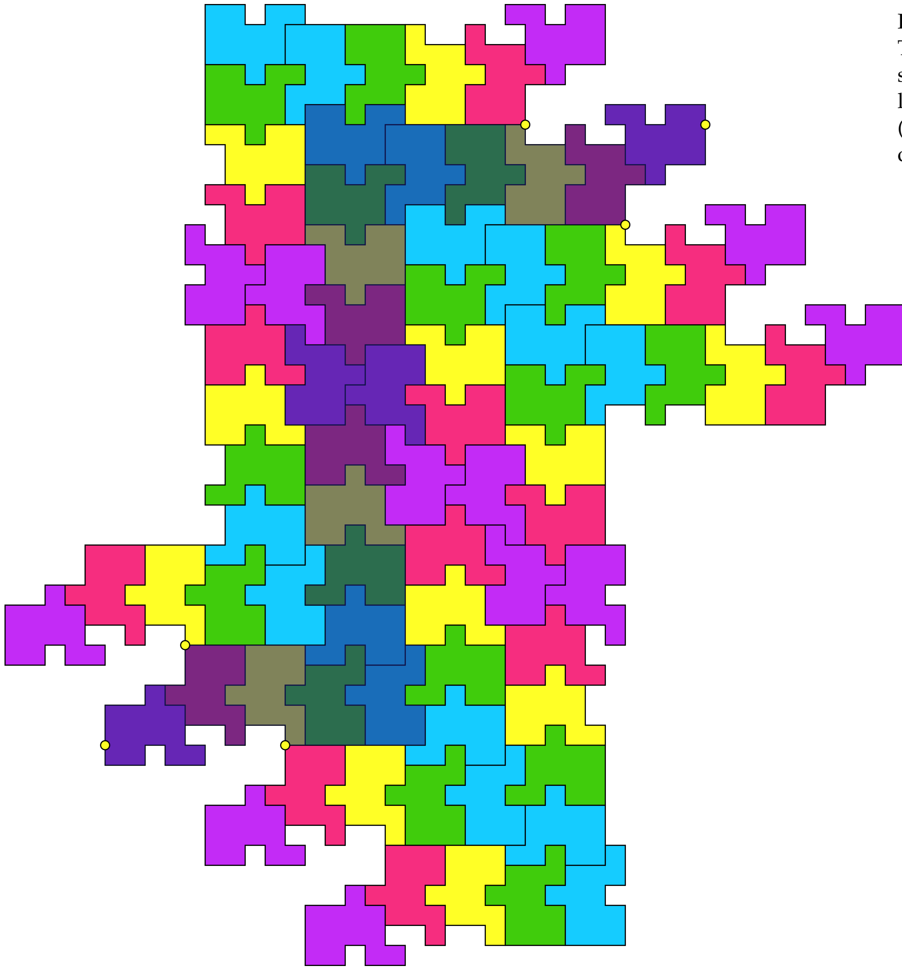


Figure 239: An 5-anisohedral tile. Transitivity classes are shown in the same color. This polyomino has the largest known fundamental region (among polyominoes) that consists out of 20 tiles. (Myers)

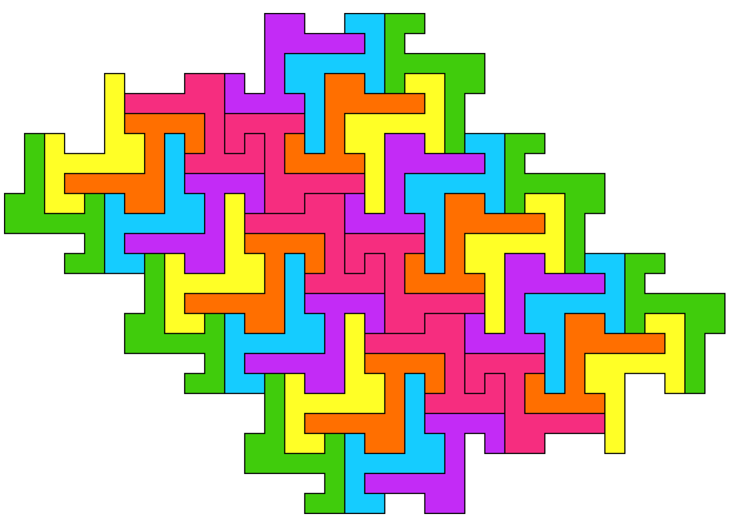


Figure 240: An 6-anisohedral tile. Transitivity classes are shown in the same color. (Myers)

Problem 62. *Find the number of transitivity classes for the tiling in Fig 241.*

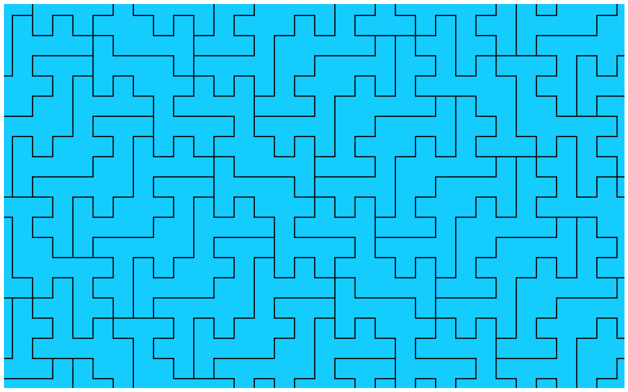


Figure 241: How may transitivity classes does this tiling have?

In all the examples found, there is one tile per transitivity class in the fundamental region. Whether this is always the case is not known. Only one example is known for a non-polyomino (shown in Figure 242).

There are essentially 9 ways a arbitrary tile can tile the plane, but two of these — Criteria 7 and 8 — cannot be realized with polyominoes, as the following theorem shows.

Theorem 245. *An equilateral triangle cannot have all it's vertices on lattice points.*

[Not referenced]

Proof. The area A of an equilateral triangle is given by $A = \frac{\sqrt{3}}{4}s$, where s is the length of the side and given by $s = \sqrt{(a-a')^2 + (b-b')^2}$. Therefor, the area is given by

$$A = \frac{\sqrt{3}\sqrt{(a-a')^2 + (b-b')^2}}{4}.$$

This value is irrational if a, a', b and b' are all integers (see Problem 63).

But by Pick's Theorem (Theorem 14), the area of any polygon with it's vertices on the lattice is rational. This is a contradiction, and therefor the equilateral triangle cannot have all its vertices on lattice points. □

Problem 63. *Prove that if the sum of two squares is divisible by 3, it is divisible by 9. (It follows that $\sqrt{3} \cdot \sqrt{x^2 + y^2}$ cannot be ration is x and y are rational.)*

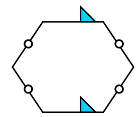
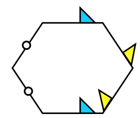
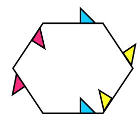
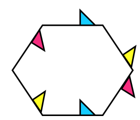
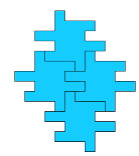
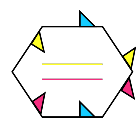
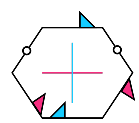
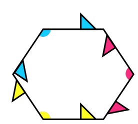
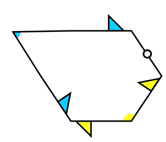
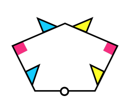
Criterion	Rules	Conceptual Diagram	Polyomino Example
Criterion 1: Conway's Criterion	(1) Sides a and d will be translates of each other. (2) Sides b, c, e, f will be centrosymmetric.		
Criterion 2	(1) Sides a and d will be translates of each other. (2) Sides b and c will be centrosymmetric. (3) Sides e and f will glide reflect to each other.		
Criterion 3	(1) Sides a and d will be translates of each other. (2) Sides b and c will glide reflect to each other. (3) Sides e and f will glide reflect to each other.		
Criterion 4: Translation Criterion	(1) Sides a and d will be translates of each other. (2) Sides b and e will be translates of each other. (3) Sides c and f will be translates of each other.		
Criterion 5	(1) Sides a and d will be translates of each other. (2) Sides b and f will glide reflect to each other. (3) Sides c and e will glide reflect to each other. (4) The glide reflections above will be parallel.		
Criterion 6	(1) Sides a and d will glide reflect to each other. (2) Sides b and f will glide reflect to each other. (3) Sides c and e will be centrosymmetric. (4) The glide reflections above will be perpendicular.		
Criterion 7	(1) Side c will rotate 120° around D to side d. (2) Side e will rotate 120° around F to side f. (3) Side a will rotate 120° around B to side b.		
Criterion 8	(1) Side c will rotate 120° around D to side d. (2) Side e will be centrosymmetric. (3) Side a will rotate 60° around B to side b.		
Criterion 9	(1) Side b will rotate 90° around C to side c. (2) Side a will be centrosymmetric. (3) Side d will rotate 90° around E to side e.		

Table 37: The 9 ways a tile can tile the plane.

n	n-ominoes	Holes	Translation	180°	Isohedral	Anisohedral	Non-tilers
1	1	0	1	0	0	0	0
2	1	0	1	0	0	0	0
3	2	0	2	0	0	0	0
4	5	0	5	0	0	0	0
5	12	0	9	3	0	0	0
6	35	0	24	11	0	0	0
7	108	1	41	60	3	0	3
8	369	6	121	199	22	1	20
9	1285	37	213	748	80	9	198
10	4655	195	522	2181	323	44	1390
11	17073	979	783	5391	338	108	9474
12	63600	4663	2712	17193	3322	222	35488
13	238591	21474	3179	31881	3178	431	178448
14	901971	96496	8672	85942	13590	900	696371
15	3426576	425449	16621	218760	43045	1157	2721544
16	13079255	1849252	37415	430339	76881	2258	10683110
17	50107909	7946380	48558	728315	48781	1381	41334494
18	192622052	33840946	154660	2344106	551137	7429	155723774
19	742624232	143060339	185007	3096983	93592	5542	596182769
20	2870671950	601165888	573296	9344528	2190553	18306	2257379379
21	11123060678	2513617990	876633	17859116	3163376	22067	8587521496
22	43191857688	10466220315	1759730	31658109	3542450	47849	32688629235
23	168047007728	43425174374	2606543	49644736	1065943	10542	124568505590
24	654999700403	179630865835	8768743	172596719	39341178	202169	475147925759
25	2557227044764	741123699012	10774339	228795554	31694933	28977	1815832051949

Table 38: Table showing counts for the number of various polyominoes that tile a certain way. From Myers.

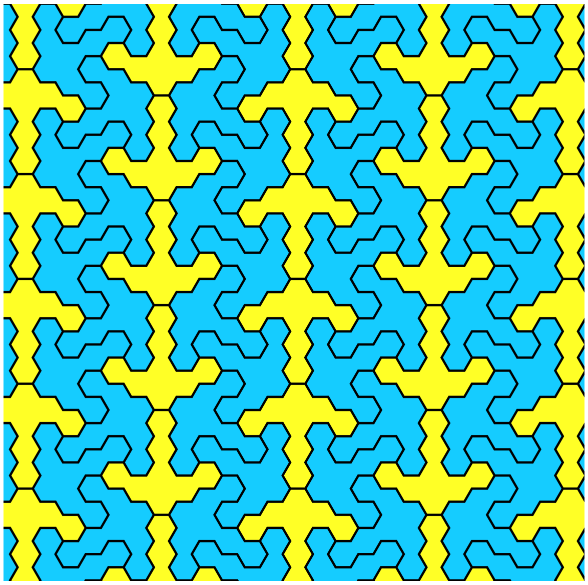


Figure 242: An example of a 2-anisotropic tile where the transitivity classes are not equally big. The tiling was discovered by Myers, and Berglund.

Table 39: Table showing number of polyominoes that are k-anisohedral Myers.

n	k = 2	3	4	5	6
8	1	0	0	0	0
9	8	0	1	0	0
10	41	3	0	0	0
11	89	18	1	0	0
12	214	6	2	0	0
13	406	24	0	1	0
14	874	24	1	0	1
15	1107	49	1	0	0
16	2210	46	1	0	1
17	1316	60	2	0	3
18	7380	42	7	0	0
19	5450	85	2	0	5
20	18211	86	5	0	4
21	21866	199	2	0	0
22	47702	135	9	1	2
23	10390	149	3	0	0
24	201834	324	11	0	0
25	28784	182	8	1	2

n	180° as well	Translation only
1	1	0
2	1	0
3	2	0
4	5	0
5	9	0
6	24	0
7	41	0
8	121	0
9	212	1
10	520	2
11	773	10
12	2577	135
13	3037	142
14	8081	591
15	13954	2667
16	32124	5291
17	41695	6863
18	118784	35876
19	150188	34819
20	411484	161812
21	604304	272329
22	1305265	454465
23	1954823	651720
24	5326890	3441853
25	7331606	3442733

Table 40: Polyominoes that tile the plane by translation Myers

7.4 m-morphic and p-poic Polyominoes

We may also ask in how many ways can a tile tile the plane. If we allow all transformations of a tile, a tile that is m-**morphic** tiles the plane in m distinct ways. If we only allow translations and rotations, we tile is p-**poic** if it tiles the plane in p distinct ways. If $m = \infty$, then we call the tile polymorphic; if $p = \infty$, we call the tile polypoic. We know tiles for all values of $m \leq 10$ (Martin, 1991, p. 96).

Table 41 gives m and p values for the various polyominoes that tile the plane.

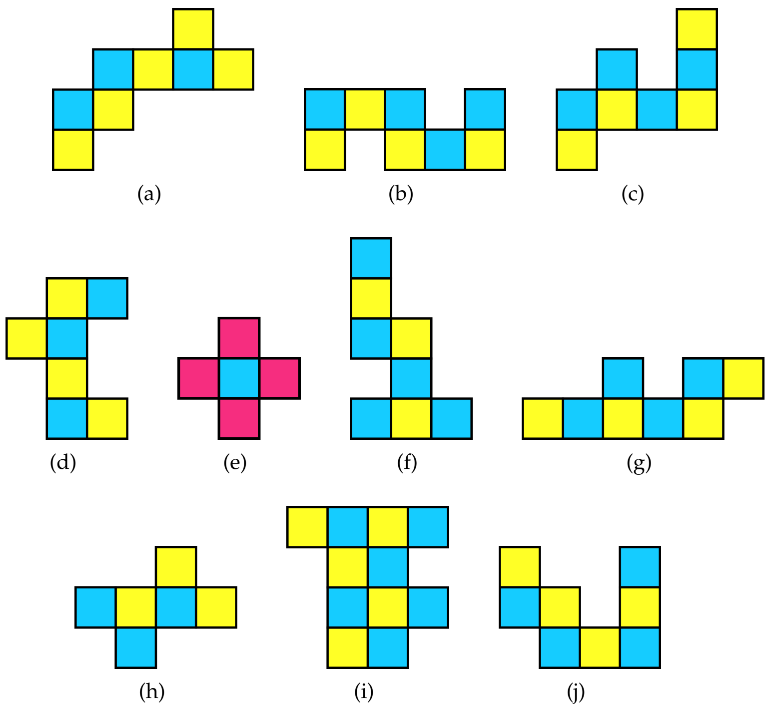


Figure 243: Diagrams of the polyominoes of Table 41.

Theorem 246. *A polyomino is hypermorphic when it tiles two prime strips of different width, or one strip in two different ways, or the period of the strip is larger than 1.*

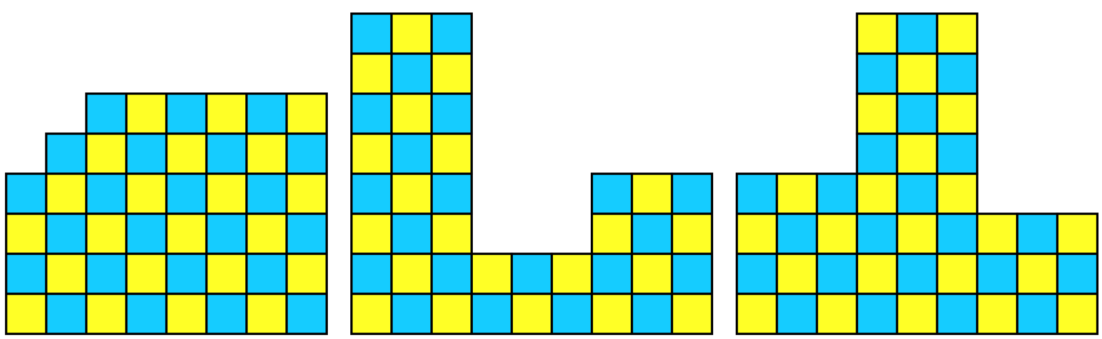
[Not referenced]

This means interesting cases involve polyominoes that belong characteristically to **PP** (or **AP** if such polyominoes exist).

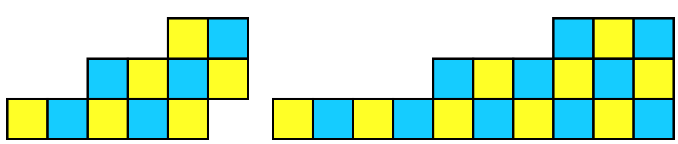
Tile	m	t	p	h
$B(1 \cdot 2 \cdot 1^2 \cdot 2)$	1	0	0	12
Figure 243 (a)	1	0	0	14
$B(1 \cdot 2 \cdot 1^2 \cdot 2 \cdot 1)$	1	1	1	16
Figure 243 (b)	1	1	1	18
Figure 243 (c)	2	1	1	∞
$B(1 \cdot 2 \cdot 1^3 \cdot 2)$	2	1	1	10
Figure 243 (d)	∞	1	1	12
Figure 243 (e)	1	2	1	13
$B(1^2 \cdot 4 \cdot 1)$	2	2	2	14
Figure 243 (f)	3	2	2	∞
$B(2 \cdot 1^2 \cdot 3)$	3	2	2	6
Figure 243 (g)	4	2	2	12
Figure 243 (h)	∞	2	2	13
$Z(2, 1, 1, 2)$	2	3	2	16
$B(1 \cdot 2 \cdot 1 \cdot 3)$	3	3	3	19
Figure 243 (i)	4	3	3	∞
$Z(2, 3, 1, 1)$	∞	3	3	14
$B(1 \cdot 5 \cdot 1)$	2	4	2	18
Figure 243 (j)	4	4	4	∞
$B(3 \cdot 2 \cdot 10)$	5	4	4	8

Tile	m	t	p	h
$B(2 \cdot 3 \cdot 4 \cdot 5^5)$	6	4	4	12
$B(4 \cdot 5 \cdot 6^6)$	7	4	4	14
$B(4 \cdot 5 \cdot 6^7)$	8	4	4	16
$B(4^3 \cdot 2^3 \cdot 8^3)$	9	4	4	18
$B(1 \cdot 2 \cdot 4 \cdot 5^2)$	∞	4	4	∞
$B(1 \cdot 8^5 \cdot 4^5)$	5	5	5	10
$B(4^3 \cdot 1^4 \cdot 9^4)$	6	5	5	12
$B(4 \cdot 21 \cdot 6)$	7	5	5	13
$B(2^2 \cdot 1^2 \cdot 7^2)$	7	5	5	14
$B(2 \cdot 1^2 \cdot 4^2)$	∞	5	5	∞
$B(3 \cdot 2 \cdot 3)$	3	6	3	6
$B(4^4 \cdot 28^4 \cdot 21^4 \cdot 14^4 \cdot 7^4)$	6	6	6	12
$B(4^4 \cdot 15^4 \cdot 10^4 \cdot 5^4)$	7	6	6	13
$B(2^2 \cdot 10^2 \cdot 5^2)$	8	6	6	16
$B(3^3 \cdot 8^3 \cdot 4^3)$	10	6	6	19
$Z(22, 1, 1)$	∞	6	6	∞
$B(1^2 \cdot 2 \cdot 3 \cdot 4)$	7	7	7	14
$B(7^2 \cdot 14^2 \cdot 8^2)$	9	7	7	18
$B(4^3 \cdot 1^4 \cdot 6^4)$	∞	7	7	∞
$B(4 \cdot 5 \cdot 4)$	4	8	4	8

Table 41: Examples of m-morphic and p-poic polyominoes. From Martin (1986).



(a) An 8-morphic polyomino. (b) A 9-morphic polyomino. (c) A 10-morphic polyomino.



(d) A 10-morphic polyomino. (e) A 11-morphic polyomino.

Figure 244: Examples of polymorphic polyominoes. The first three is from Winslow (2018, Figure 8), the last two from Myers.

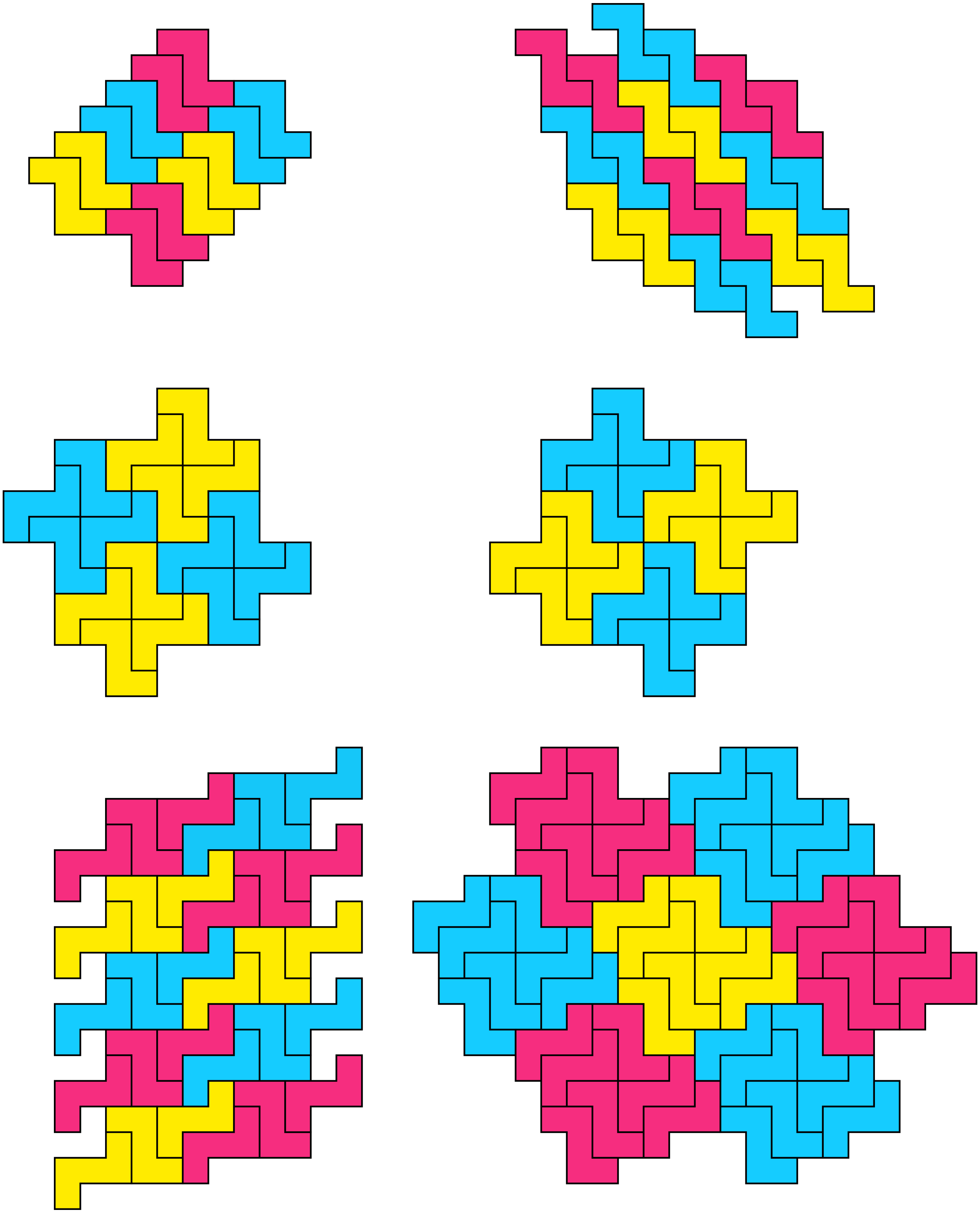


Figure 245: The 6 tilings of the Z-pentomino if reflection is not allowed.

					m								
n	1	2	3	4	5	6	7	8	9	10	11	∞	Skipped
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	0	0	0	0	0	2	0
4	0	0	0	0	0	0	0	0	0	0	0	5	0
5	1	0	0	0	0	0	0	0	0	0	0	11	0
6	0	0	0	0	0	0	0	0	0	0	0	35	0
7	6	6	3	0	0	0	0	0	0	0	0	89	0
8	20	18	4	2	0	0	1	0	0	0	0	298	0
9	193	84	14	6	1	0	0	0	0	0	0	752	0
10	749	257	41	2	2	1	0	0	0	0	0	2018	0
11	3222	809	148	31	12	3	2	0	0	1	0	2392	0
12	9026	1440	153	22	4	0	0	0	0	0	0	12803	1
13	25090	3645	435	94	25	4	3	3	0	0	0	9368	2
14	63746	5681	416	56	17	2	1	0	0	0	0	39176	9
15	180669	11842	665	85	13	1	0	1	0	0	0	86306	1
16	366557	16758	1128	142	37	6	2	0	0	0	0	162257	6
17	683157	30733	1473	264	63	9	2	1	0	0	0	111321	12
18	2192816	65557	2226	238	31	1	1	1	0	0	0	796444	17
19	2936540	72811	2130	360	96	7	2	1	1	1	1	369160	14
20	9444080	126363	3550	322	68	42	1	1	0	0	0	2552238	18
21	18299457	221446	4767	458	90	14	0	0	2	0	0	3394939	19

Table 42: The number of polyominoes that are m-morphic Myers. (Meyers originally classified the monomino as monomorphic—presumably because he allows non-discrete tilings.)

7.5 *Non Tilers*

We call a polyomino that does not tile the plane a **non-tiler**. The smallest non-tilers have 7 cells; there are only 3 without holes, shown in Fig. 246. The 20 simply-connected octominoes that don't tile the plane is shown in Fig. 247. Numbers for non-tiles of polyomninoes with 25 cells or less is shown in Table 38.

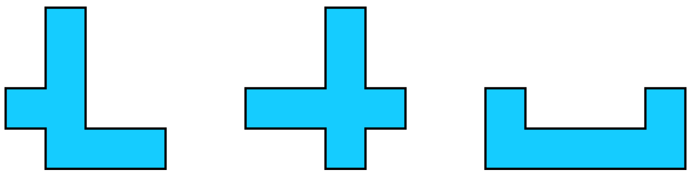


Figure 246: Heptominoes without holes that don't tile the plane.

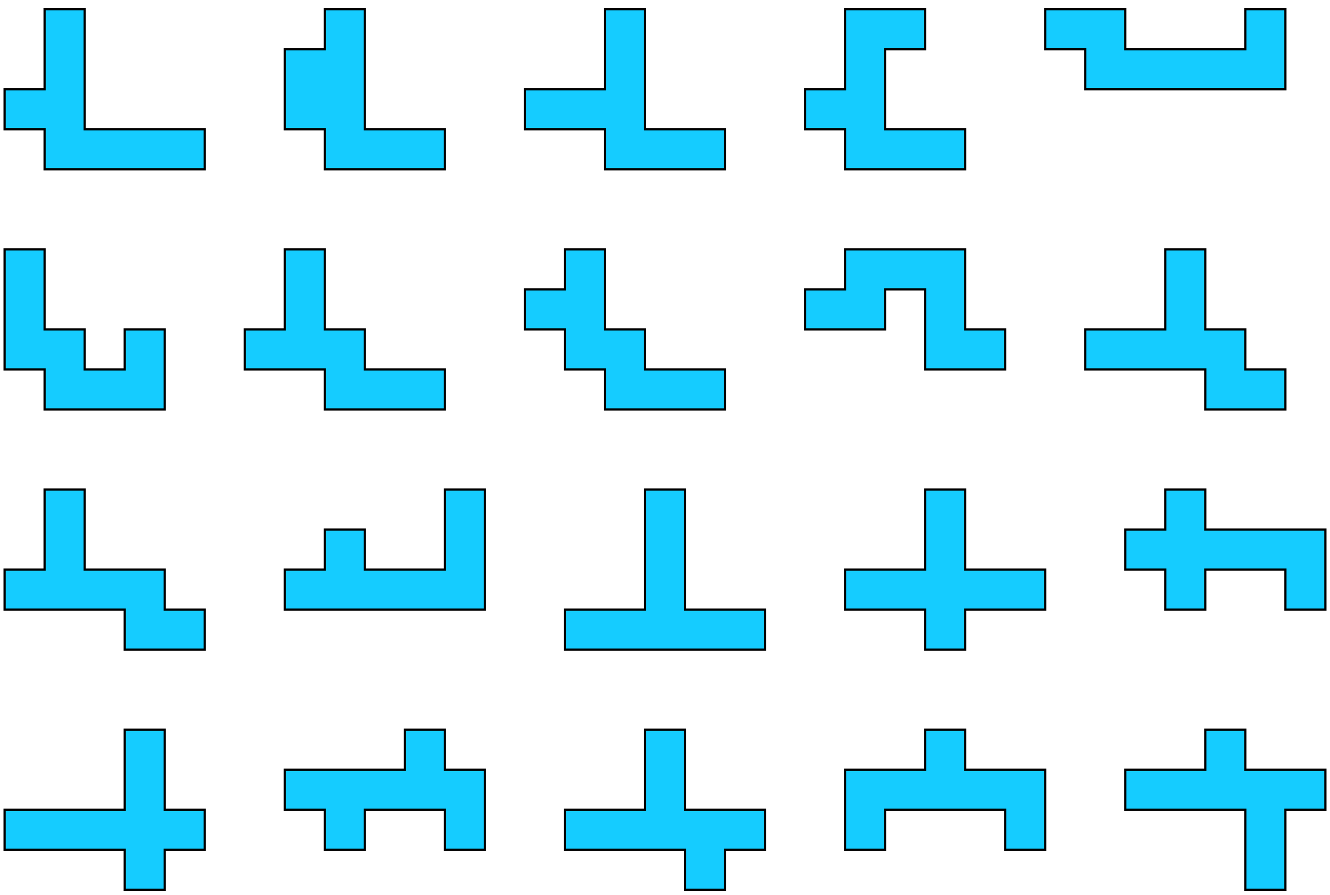


Figure 247: Octominoes without holes that don't tile the plane.

When a tile does not tile the plane, we may wonder: how close can it come to tiling it?

One way to make this question precise is to ask "How many rings of a tile can be place around another tile so that there are no holes?"

The answer to this question is called a tile's **Heesch number**. Currently, we don't know of any polyomino that does not tile the plane with a Heesch number greater than three; in fact, we do not know of *any* tile with Heesch number bigger than 5 (see Mann (2004) for examples).

Figure 249 shows all the known polyominoes with Heesch number 2, with some example tilings in Figure 248. The polyomino in Figure 250 has Heesch number 3; is derived from the *4-polypillar* in Mann (2004) and Mann and Thomas (2016).

Some results for polyominoes are shown in Table 43.

		Heesch number			
n	Hole-free non-tilers	0	1	2	Unclassified
7	3	1	2		
8	20	6	14		
9	198	77	120	1	
10	1390	770	620		
11	9474	6029	3409	26	10
12	35488	29114	6369	5	
13	178448	152339	26025	40	44
14	696371	642259	53537	26	549

Table 43: Heesch Numbers for Polyominoes Kaplan (2017)

A **U-frame polyomino** is a polyomino of the form $x^{-h}y^{g}x^{-f}y^{-e}x^{d}y^{c}x^{-b}y^{-z}$.

Theorem 247 (Fontaine (1991)). *A U-frame polyomino has Heesch number 2 if:*

(1) $a = g + h - e$

(2) $b = g + h$

(3) $c = h$

(4) $d = 2g + 3h$

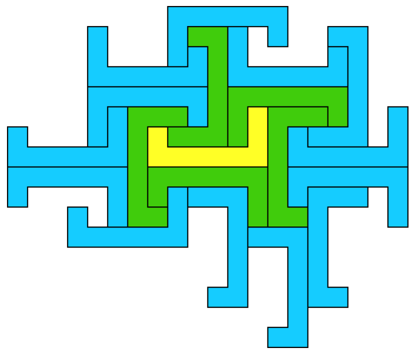
(5) $f = g + h$,

where $0 < g < e < h < 2e$,

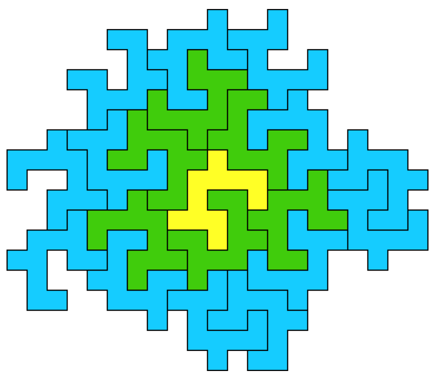
[Not referenced]

7.6 *Aperiodic Tilings*

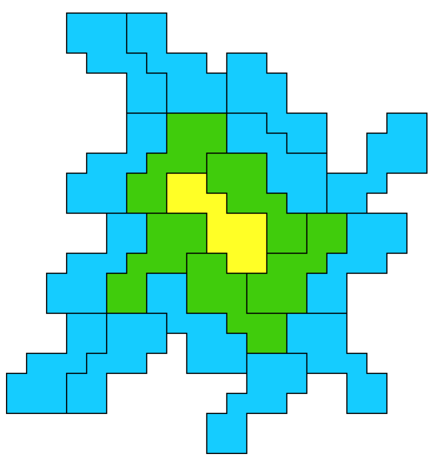
While it is easy to find tilings that are non-periodic, it is not so easy to find tile aperiodic tile sets. Two examples of aperiodic sets are shown in Figures 251 and 253. The tiling of the first set is shown in Figure 252.



(a) Kaplan (2017)



(b) Rhoads (2003)



(c)

Figure 248: Some almost-tilings of polyominoes with Heesch number 2

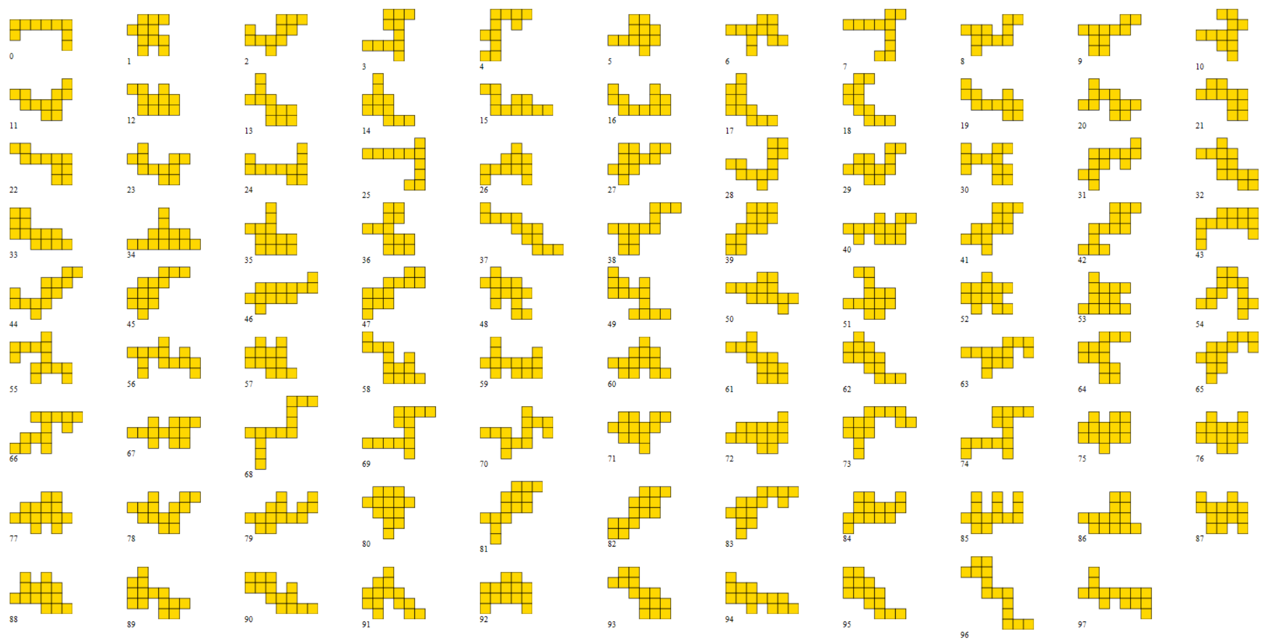


Figure 249: Polyominoes with Heesch number 2. List obtained from Kaplan (2017).

A **Wang tile**[8] is a square with colored edges; in a legal tiling of Wang tiles colors must match. Despite their simplicity, there are sets of Wang tiles that tile the plane aperiodically. Such a set must use at least four colors Chen et al. (2014), and have at least 11 tiles Jeandel and Rao (2015). An example is shown in Figure 258.

There is a way to convert between polyomino problems and Wang tile problems (Golomb, 1970). To convert from Wang tiles to polyominoes, the basic idea is to use a base tile that is monomorphic, and encode the colors in binary by perforating the edges (by adding and removing cells). For this to work, the base tile must be big enough. Figure 256 shows the sequence of G_i of base tiles introduced. To encode 4 colors, we need the perforatable edge to be 2 cells wide, so we can use G_2. The set of 11 Wang tiles realized as polyominoes is shown in Figure 259.

Another encoding scheme (from Yang (2014)) is shown in Figure 257.

To convert from polyominoes to Wang tiles, we will have a unique Wang tile for each cell in the set. We choose one color for outer edges, and one color for each internal edge. This way, the only tilings will match a polyomino tiling exactly.

Because of their simple structure, Wang tiles are much easier to work with. Because of the conversion, some general theorems that can be proven for Wang tiles also apply to Polyominoes.

Here are some examples:

[8] Golomb (1970) calls this a *MacMahon tile*, after Major P.A. MacMahon who studied their properties in MacMahon (1921). The question whether aperiodic sets of Wang tiles exist was first asked in Wang (1961). The first aperiodic tile set is a set of 20,426 Wang tiles, described in Berger (1966), where the term *Wang tile* is also used for the first time.

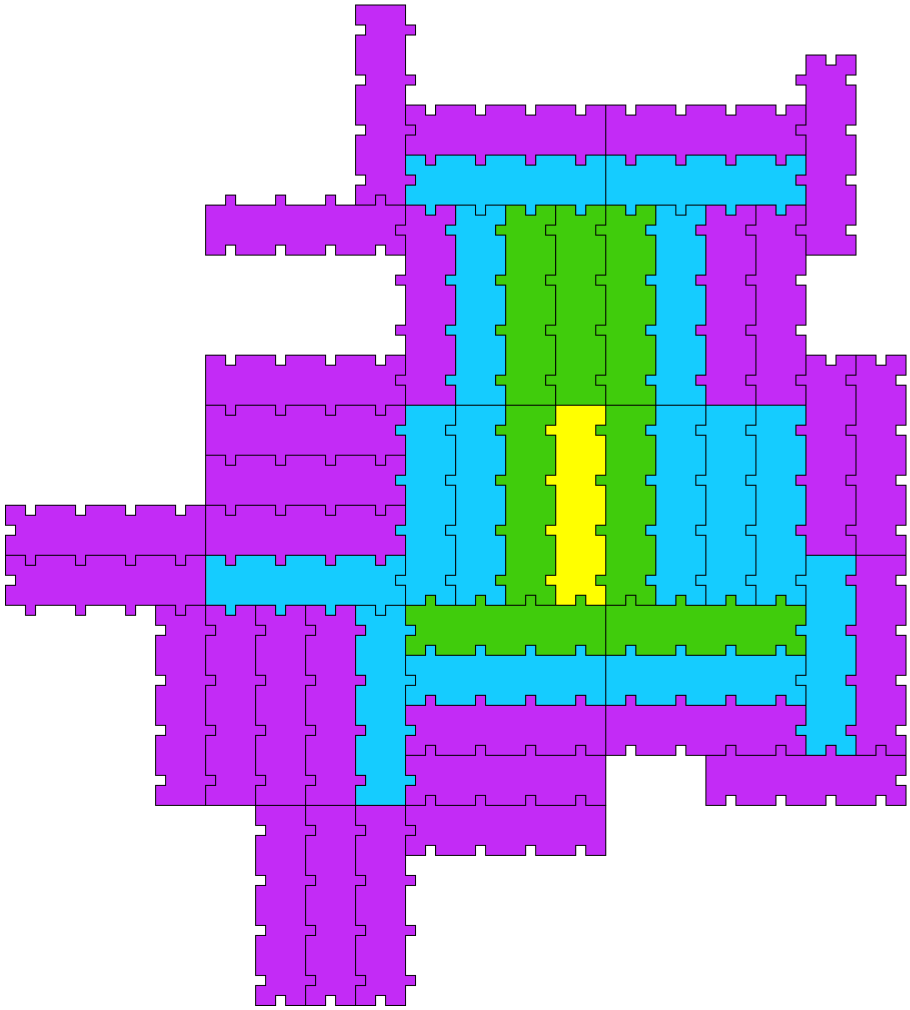


Figure 250: A 99-omino with Heesch number 3.

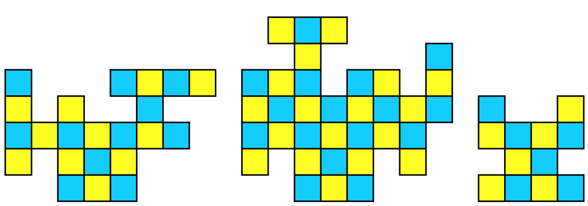


Figure 251: An aperiodic tile set discovered by Roger Penrose derived from a set by Robert Amman(Penrose, 1994, Fig. 1.3, p. 32).

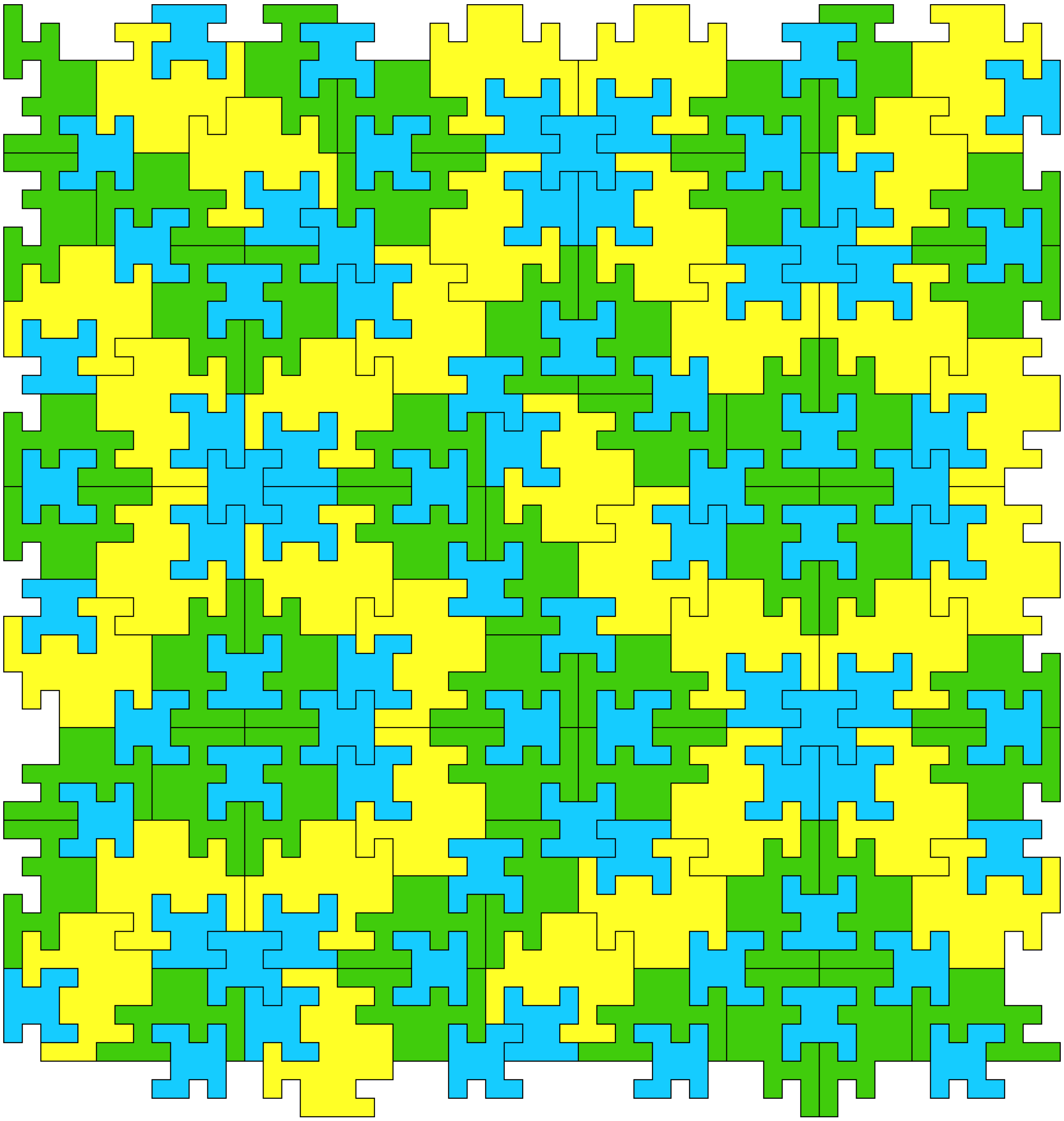


Figure 252: A patch of a plane tiling by the Penrose set.

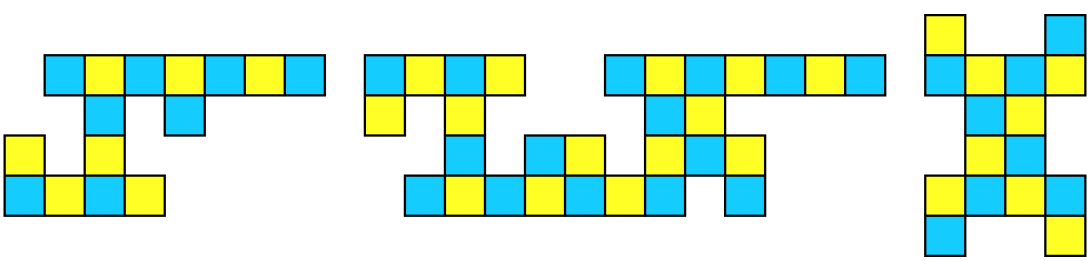


Figure 253: An aperiodic set discovered by Matthew Cook (Wolfram, 2002, p. 943).

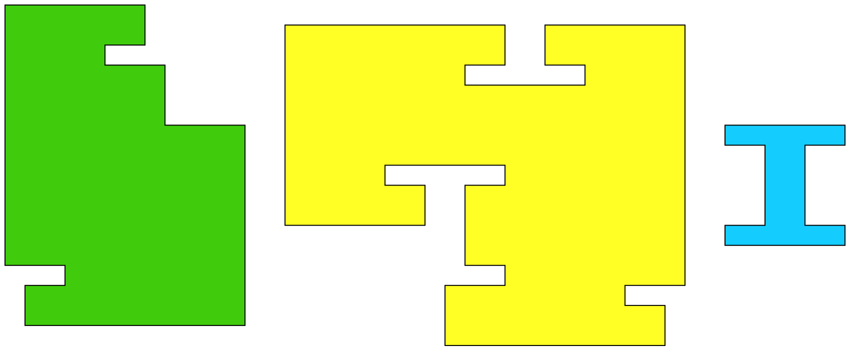


Figure 254: An aperiodic set given in (Winslow, 2015, Fig. 5), modified from a similar set in Ammann et al. (1992, Fig. 1).

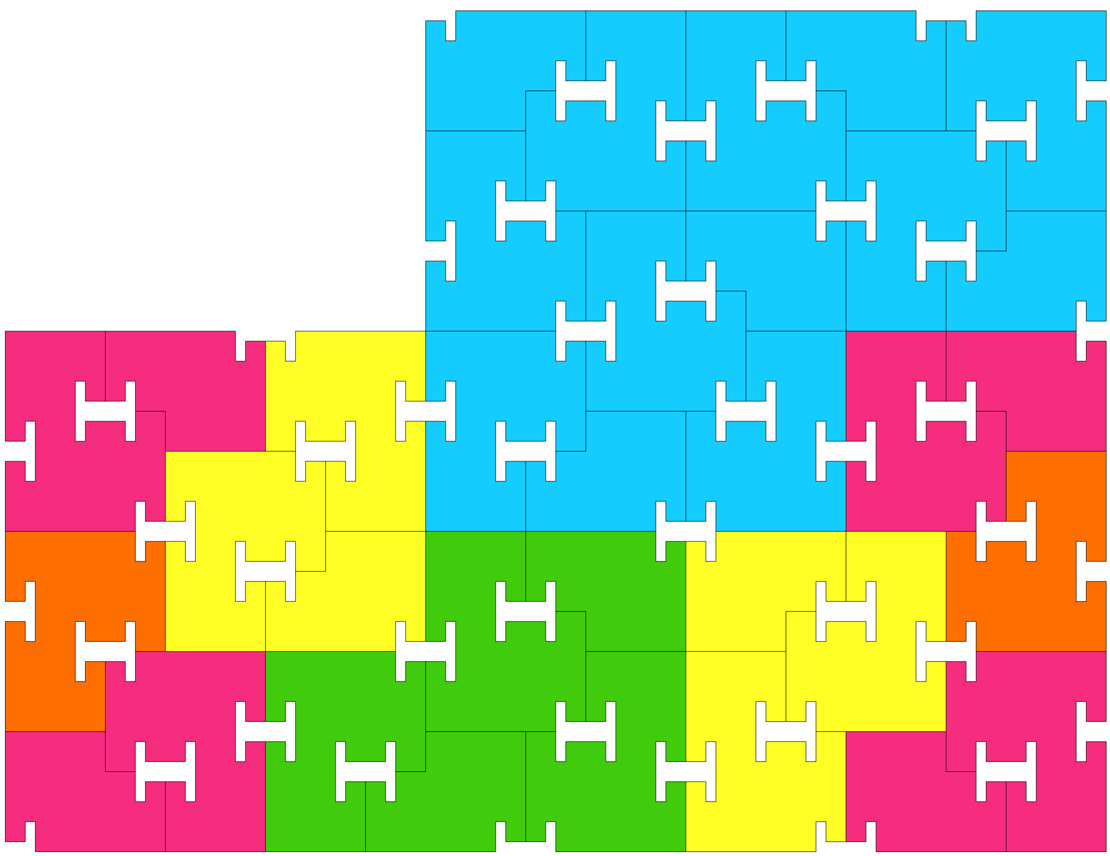


Figure 255: A tiling by Wilson's tiles.

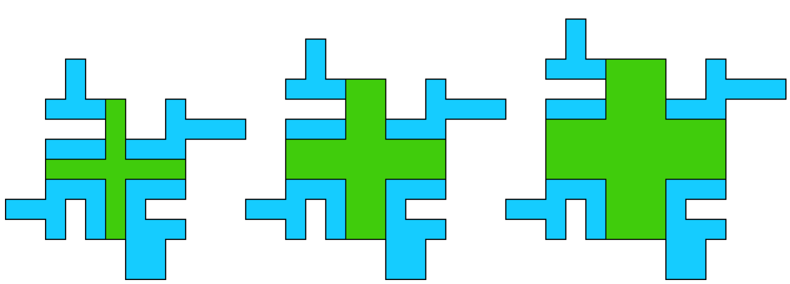


Figure 256: Golomb's Encoding of Wang tiles.

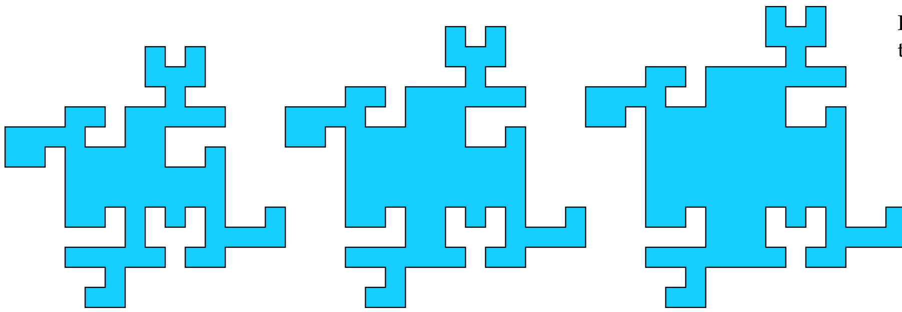


Figure 257: Yang's encoding of Wang tiles.

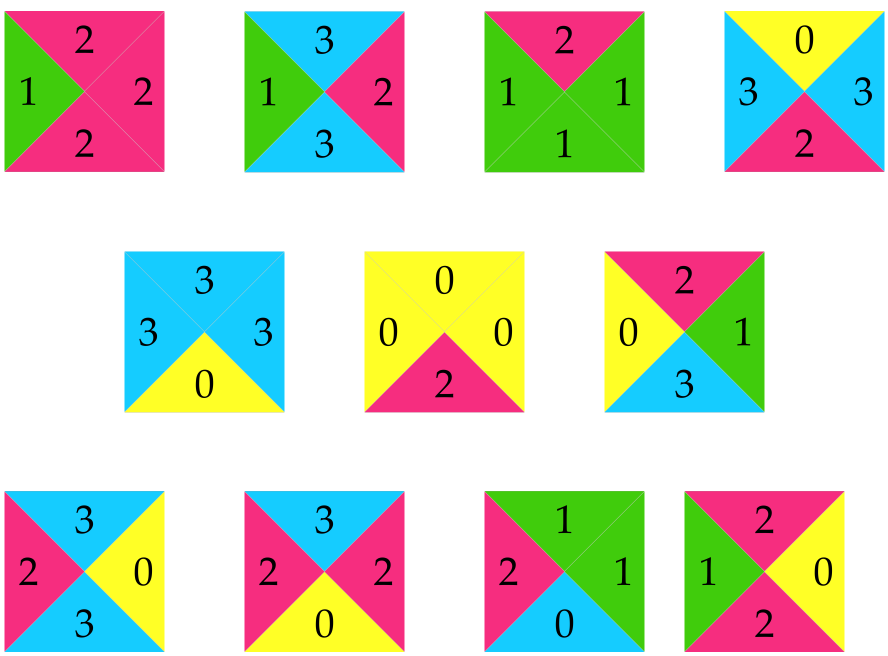


Figure 258: The smallest aperiodic set of Wang tiles.

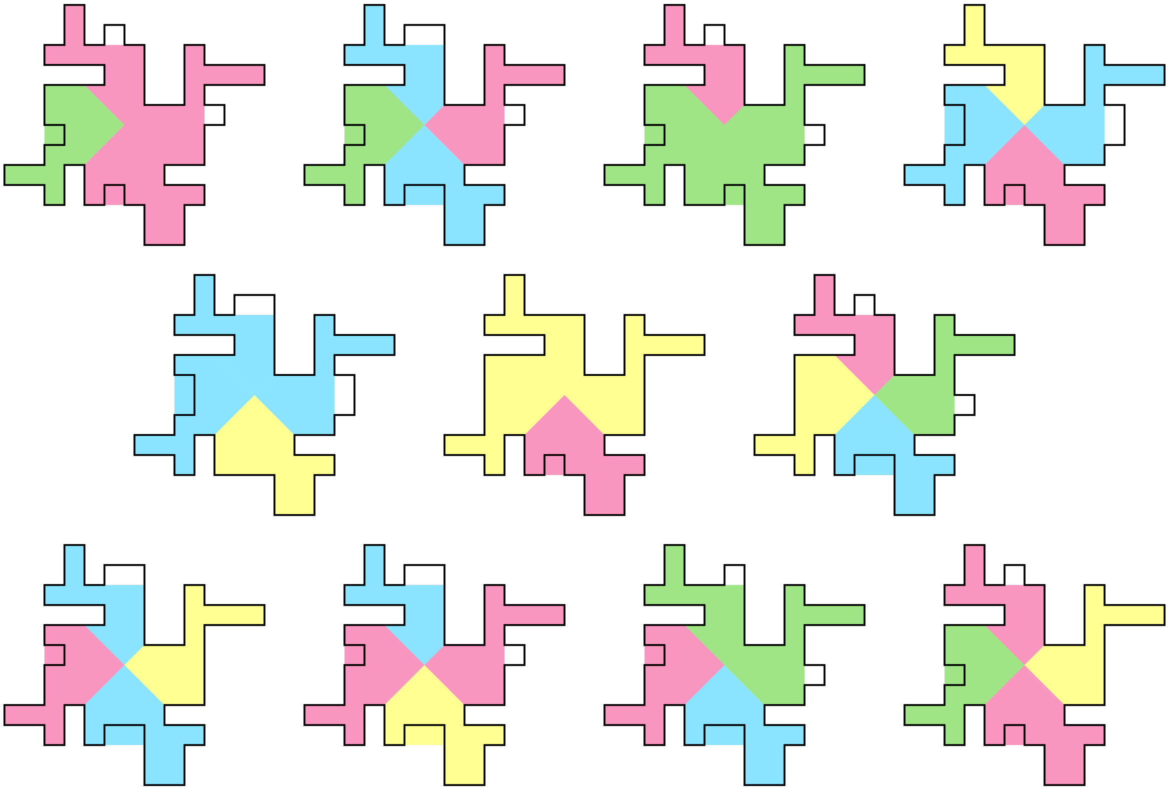


Figure 259: Encoded as polyominoes.

(1) Whether a set of Wang tiles (polyominoes) can tile the plane is undecidable.

(2) Whether a set of Wang tiles (polyominoes) can tile an infinite strip is decidable.

(3) There are no aperiodic tilings of a infinite strip.

7.7 *Further Reading*

Plane tilings for more general tiles than polyominoes is widely covered. (Toth et al., 2017, Chapter 3) is a handy reference for terminology, main results and open problems. I already mentioned Grünbaum and Shephard (1987), which is the most comprehensive source on the topic. Kaplan (2009) is a lighter read, and especially suitable if you want to use the computer to experiment. Conway et al. (2016) is a very interesting book, focusing on symmetry, algebra, and topology, not only on the plane, but also on the sphere and hyperbolic plane. Other resources include Horne (2000), and a short survey Goodman-Strauss (2016). Open problems is listed and briefly discussed in Brass et al. (2005).

The 1270 types of 2-isohedral tilings of the plane is given in Delgado et al. (1992). Huson (1993) contains information about classification of isohedral, 2-isohedraL and 3-isohedral tilings of the plane, sphere, and hyperbolic plane.

8
Various Topics II

The purpose of this section is to introduce various interesting topics and provide some references. I may expand some of these topics later.

8.1 Compatibility

Two polyominoes P and Q are **compatible** if there is a finite[1] region R that they can both tile Cibulis et al. (2002). Of course, if one polyomino tiles another, they are compatible. Any two rectangles are compatible, since $R(m, n)$ and $R(m', n')$ both tile $R(mm', nn')$ by Theorem 154. If P tiles P' and Q tiles Q', and P' is compatible with Q', then P is compatible with Q. It follows then that any two rectifiable polyominoes are compatible. The monomino is compatible with any polyomino, a property not shared even with dominoes, which is incompatible with . Minimal incompatible figures can be found in Sicherman. There are many unproven conjectures for this problem. The compatibility of pentominoes is shown in Tables 44 and 45.

[1] There are more interesting problems if we remove this restriction: what polyominoes are infinitely compatible but not finitely? Can the infinite regions that serve as common multiples be characterized?

	F	I	L	N	P	T	U	V	W	X	Y	Z
F	1	10	2	2	2	2	4	4	2	2	2	2
I	10	1	2	2	2	4	12	4	10	×	2	20
L	2	2	1	4	2	2	2	2	2	44	2	2
N	2	2	4	1	2	2	2	2	2	16	2	2
P	2	2	2	2	1	2	2	2	2	4	2	2
T	2	4	2	2	2	1	4	2	14	4	2	2
U	4	12	2	2	2	4	1	2	2	×	2	4
V	4	4	2	2	2	2	2	1	6	×	2	4
W	2	10	2	2	2	14	2	6	1	?	2	4
X	2	×	44	16	4	4	×	×	?	1	2	?
Y	2	2	2	2	2	2	2	2	2	2	1	2
Z	2	20	2	2	2	2	4	4	4	?	2	1

Table 44: Table showing the least number the number of tiles necessary to tile the LCM (as far as is known). From Sicherman.

- Self
- Overall compatibility unknown
- Overall incompatible

	F	I	L	N	P	T	U	V	W	X	Y	Z
F	1	10	2	2	2	2	4	6	2	2	2	2
I	10	1	2	2	2	32	?	10	10	×	2	?
L	2	2	1	4	2	2	2	2	2	×	2	2
N	2	2	4	1	2	2	2	2	2	16	2	2
P	2	2	2	2	1	2	2	2	2	4	2	2
T	2	32	2	2	2	1	?	2	16	4	2	30
U	4	?	2	2	2	?	1	?	2	×	2	?
V	6	10	2	2	2	2	?	1	6	×	2	4
W	2	10	2	2	2	16	2	6	1	?	2	10
X	2	×	×	16	4	4	×	×	?	1	2	?
Y	2	2	2	2	2	2	2	2	2	2	1	2
Z	2	?	2	2	2	30	?	4	10	?	2	1

Table 45: Table showing the least number the number of tiles necessary to tile the LCM (as far as is known). From Sicherman.

- Self
- Overall compatibility unknown
- Holeless compatibility unknown
- Overall incompatible
- Holeless incompatible.
- Overall best solution has holes.

Also of interest is the set of regions that can be tiled by each of a set of polyominoes, called a **common multiple**; a smallest region that is a common multiple of some polyominoes is called a **smallest common multiple** (Cibulis et al., 2002). The smallest common multiple of the tetrominoes is shown in Figure 260.

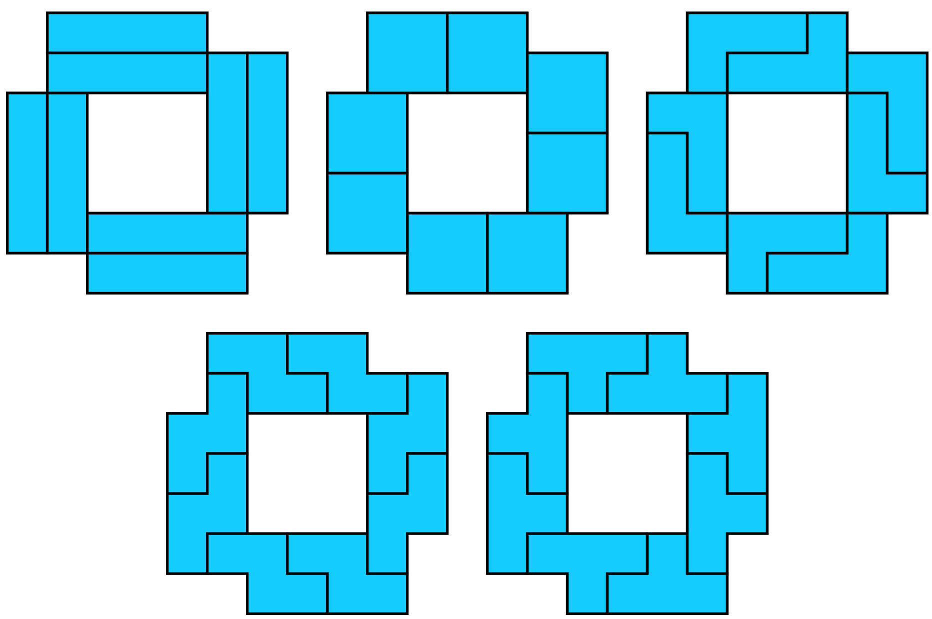


Figure 260: The smallest (known) common multiple of the tetrominoes.

The basic idea is in a slightly more general setting in Golomb (1981). For more, see Barbans et al. (2003), Cipra et al. (2004), Pegg et al. (2009), and Liu (2018b). Large lists of compatibility tables can be found on Resta and Sicherman.

8.2 Exclusion

What is the least number of monominoes we can place inside $R(m, n)$ so we cannot place a polyomino P anywhere? What fraction of cells of the plane must be covered by monominoes so that a polyomino P cannot be placed anywhere? This problem is introduced in Golomb (1996, Chapter 3).

Theorem 248. *If P is an efficient polyomino, it can be excluded from the plane with* $1/|P|$ *monominoes.*

[Not referenced]

Proof. Suppose K is an optimal coloring for P. Then put a monomino wherever $K(x, y) = 0$. Since however we place a polyomino it must cover each color once, if there is no cells with color 0 available, it is impossible to place P anywhere. □

Among others, this gives us colorings for various rectangles (all those whose one side divides the other, including squares and bars) by Theorem **??**.

Problem 64. *Is the converse true; that is: if we can exclude a polyomino P from the plane with* $1/|P|$ *monominoes, is P efficient?*

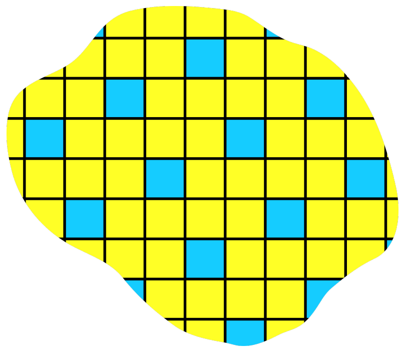
The following allow us to get a lower bound on the exclusion fraction. Suppose a certain arrangement excludes P, and $P \subset Q$. Then Q is also excluded by the same arrangement.

Problem 65. *Find exclusion patterns for the hexominoes.*

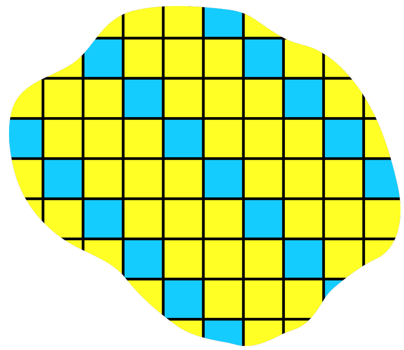
	n								
m	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	5	8	8	12	12	16	16
3	3	5	9	10	13	18	18	18	27
4	4	8	10	16	17	17	25	32	32
5	5	8	13	17	25	26	26	26	41
6	6	12	18	17	26	36	37	37	37
7	7	12	18	25	26	37	49	50	50
8	8	16	18	32	26	37	50	64	65
9	9	16	27	32	41	37	50	65	91

Table 46: Inverse frequency for rectangles. (Adapted from Klamkin and Liu (1980))

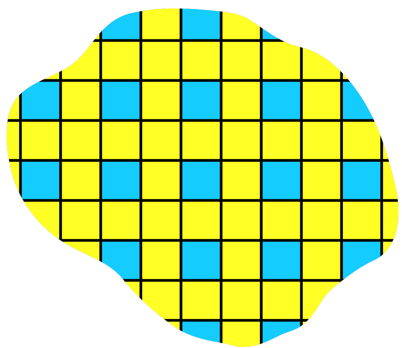
This exclusion problem for pentominoes on rectangles has been discussed in Gravier and Payan (2001) and Gravier et al. (2007), and extended to other types of graphs. Klamkin and Liu (1980), Barnes and Shearer (1982) treats the exclusion problem on the plane.



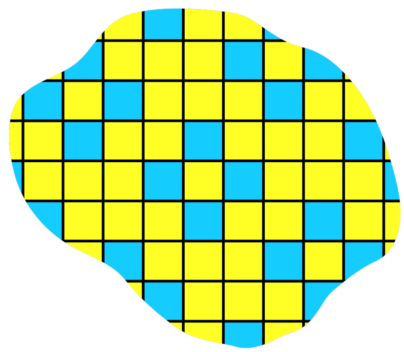
(a) Excludes X, I ($F_{5,2}$)



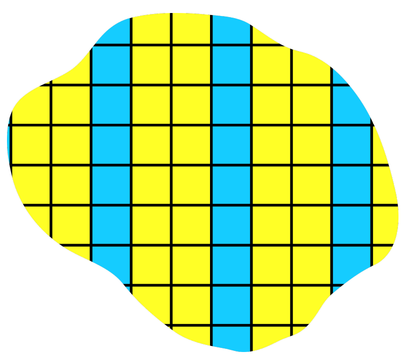
(b) Excludes L, T, U, Y (F_4)



(c) Excludes F, N, P, W (S_2)



(d) Excludes V ($F_{13,8}$)



(e) Excludes Z($F_{3,0}$)

Figure 261: Exclusion patterns for pentominoes. The pattern can be obtained from the coloring in brackets by putting a monomino at each cell with color 0. From Golomb (1996, Fig. 67), Klamkin and Liu (1980, Fig. 3).

8.3 *Fountain sets*

The monomino tiles any polyomino. Are there other sets of polyominoes that tile "almost all" polyominoes?

The set in Figure 262 tiles all polyominoes, except for the monomino.

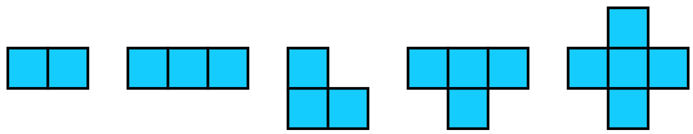


Figure 262: A fountain set. These polyominoes can tile all polyominoes, except the monomino.

In general, a **fountain set** is a set of polyominoes that can tile all polyominoes formed from appending a monomino to a member of the setKenyon and Tassy (2015).

Fountain sets are often infinite. In fact, any fountain set that contains no bars is infinite.

We can extend this idea in several ways:

- What smallest sets of polyominoes can tile all but a finite set of polyominoes? polyominoes that have area divisible by n? polyominoes that are balanced for some coloring K?
- What sets can tile all elements of the set appended with a domino? some other polyomino? some other set of polyominoes?

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