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DOMINO TILINGS I

V.1.1, 1 MAY 2018

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Preface and Introduction

Preface v1.0

A FEW MONTHS AGO I started on a journey to discover polyominoes and especially their tilings. I wanted to collect and organize what I found on this journey, and started to put it into a few essays. This is the first.

These essays are meant to be accessible, so they don't require a lot of algebra, graph theory, combinatorics, or other machinery that is often used in modern techniques. At times this means I cover only a very concrete special case of a more general (and arguably beautiful) result, and some proofs in their naive disguise may be a bit clunky; such is the tradeoff.

That said, I try to go beyond the classical presentation of Golomb (1996) and Martin (1991). To give one example, the first essay on dominoes covers more than simply the checkerboard coloring—we look at *flow*, the *marriage theorem*, *geometric transformations*, and a lot more.

This is a first draft; I decided to post it on my blog to get some early feedback rather than spend months polishing everything to perfection. If you have any comments, or notice an error or missing reference, please let me know[1].

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Acknowledgments

I want to thank Justin Southey for reading a draft of this essay. He made many suggestions that improved it greatly.

A Quick Introduction to Polyominoes and Tilings

POLYOMINOES ARE SHAPES formed by "gluing" 1×1 squares together, edge-to-edge. Figures 1 to 6 show the polyominoes with five cells or less. The squares that make up a polyomino are called **cells**,

and we classify polyominoes according to the number of cells they have. The names for the smaller classes of polyominoes is given in Table 1.

n	Name
1	Monomino
2	Domino
3	Tromino
4	Tetronino
5	Pentomino

Table 1: Names for classes of polyominoes

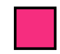


Figure 1: Monomino

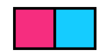


Figure 2: Domino

Figure 3: Bar tromino

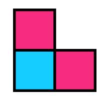


Figure 4: Right tromino

These simple shapes give rise to a variety of interesting problems. The one which will be the main topic of these essays, is: Which shapes can we build if we put them together so that edges line up?

Problems such as these are called *tiling problems*. A different way to put a tiling problem is: given some shape (we call this a **region**) and a set of other shapes (called the **tiles**), can we cover the region completely with copies of the tiles so that no piece of tile falls outside the region, no tiles overlap? If this is possible, we say the region can be tiled by the set of tiles.

Tiling is a big and difficult topic in mathematics[2], but we can get some insight by limiting our study to polyominoes.

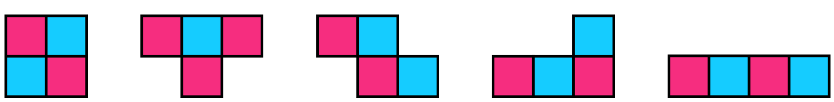
[2] The magnificent Grünbaum and Shephard (1987) covers the topic in depth. For a lighter overview, see Ardila and Stanley (2010).

In our case, we will only work with regions that are also composed from squares glued edge-to-edge. Regions are often polyominoes themselves, but we will at times also consider infinite regions, or disconnected regions, or regions with barriers that tiles are not allowed to cross.

The number of cells of a finite region R is denoted by $|R|$. Since each cell is has an area of 1, $|R|$ also equals the area.

In a region or a polyomino, we call two cells **neighbors** if they share an edge. A cell with four neighbors is called an **interior cell**; otherwise it is called a **border cell**. A cell with only one or two neighbors is called a **corner cells**.

Two cells are **connected** if we can move from neighbor-to-neighbor from one to the other. We call a region where every cell is connected to every other cell **connected**, and if it has no holes, we call it **simply-connected**.



(a) Square (b) T (c) Skew (d) L (e) Bar

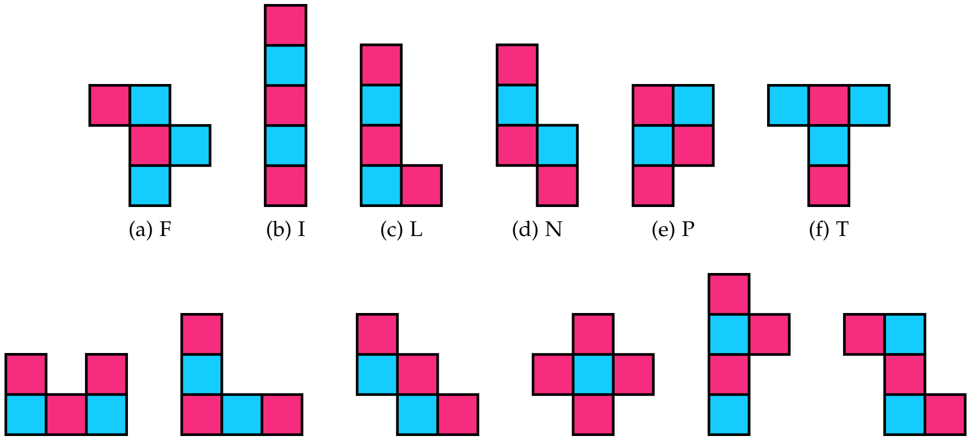
Figure 5: Tetrominoes

To conclude this quick introduction, I state two obvious theorems that we will reference often. I omit their proofs.

Theorem 1 (Area Criterion). *If we have a tile set where all tiles have area n, then we can only tile regions whose area is divisible by n.*

[Referenced on pages 9 and 12]

Example 1. *If a region is tileable by dominoes, then its area is even.*



(a) F (b) I (c) L (d) N (e) P (f) T

(g) U (h) V (i) W (j) X (k) Y (l) Z

Figure 6: Pentominoes

If S is a subset of the cells of a region R, we say S is a **subregion** of R, (see Figure 7), and if in addition $S \neq R$, S is a **proper subregion** of R. The set of cells in R not in S is denoted $R - S$. A partition of a region R is a collection of subregions S_i of R such that each subregion has at least one cell, and no two subregions have a cells in common. To **partition** a region means to give a partition of the region.

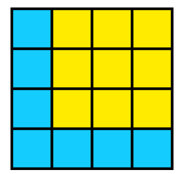


Figure 7: In this example, the 4×4 square is the region, and both the yellow and blue regions are subregions of the square.

Theorem 2 (Partitions)**.** *If all the subsregions of a partitions of a region are tileable, then so is the region.*

[Referenced on pages 33, 35 and 36]

The Geometry and Topology of a Polyomino

(Note: this section needs to be reworked completely; as it stands it has a lot of problems, especially with vague definitions that make the proofs unconvincing. I left this section in, because some of the ideas here are relevant to other sections.)

A **rectilinear polygon**[3] is a polygon with all edges meeting at right angles. A polyomino, therefor, is a rectilinear polygon whose edges are integer lengths. In this section, we prove a few simple facts about rectilinear polygons that will be useful to our study of polyominoes.

[3] Also called *orthogonal polygon*.

A figure without holes[4] is called **simply-connected**.

[4] Although it is intuitively clear what a hole is, it is a bit harder to give a precise definition, and we will not do this here.

Theorem 3. *In a simply-connected rectilinear polygon, the number of vertical and horizontal edges are equal.*

[Not referenced]

Proof. Each vertical edge is followed by a horizontal edge, and vice versa. □

Theorem 4. *In a rectilinear polygon the total horizontal edge length and total vertical edge length are even, and so is the perimeter.*

[Referenced on page 18]

Proof. □

Theorem 5. *In a simply-connected rectilinear polygon, the number of edges is even.*

[Referenced on page 18]

Proof. Since there is the same number of vertical edges as horizontal edges (since each vertical edge is followed by a horizontal edge), their sum is even. □

A corner with interior angle of 90 degrees is called **convex** and a corner with interior 270 is called **concave** Bar-Yehuda and Ben-Hanoch (1996).

Theorem 6 (Bar-Yehuda and Ben-Hanoch (1996))**.** *In a finite, simply-connected rectiliear polygon, the number of convex corners is 4 more than the number of concave corners.*

[Not referenced]

If the figure is not finite, it may not have any corners. And if we allow holes, we can have more concave corners than convex corners.

Problem 1. *Give examples of figures (not necessarily finite or simply-connected) with:*

(1) *No corners.*

(2) *One convex corner and no concave corners.*

(3) *Two convex corners and no concave corners.*

(4) *Four concave corners and no convex corners.*

Prove the following figures are impossible. Figures with:

- *Three convex corners and no concave corners.*
- *One, two, or three concave corners and no convex corners.*

Prove that all other combinations are possible.

It follows that any rectilinear polygon must have at least 4 convex corners.

An edge is called a **peak** if lies between two convex corners, a **valley** if lies between two concave corners, and a **flat segment** otherwise. 5

[5] In Bar-Yehuda and Ben-Hanoch (1996) the author calls a peak a *knob* and a valley an *anti-knob*. The term *peak* is used in Beauquier et al. (1995), and since it allows the natural extension to *valleys* and *flats*, those are the terms used in this book. Note that the notion of a k-knob (for $k = 1, 2, 3, 4$) used in Bar-Yehuda and Ben-Hanoch (1996) is completely different from the notion of n-peak (for $n \geq 1$ we will define later.

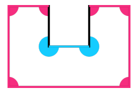


Figure 8: In this figure, convex corners are red, and concave corners are blue. The five peaks that lie between pairs of convex corners are marked red, and the valley between the pair of concave corners is blue. The two remaining (black) edges are flats.

Theorem 7. *Let P be the number of peaks and V the number of valleys of a finite simply-connected figure. Then $P - V = 4$.*

[Referenced on pages 7 and 8]

It follows that for finite simply-connected figures, the number of peaks must always be at least 4. Furthermore:

Theorem 8. *If F is the number of flats for a finite simply-connected figure, then F is even.*

[Not referenced]

Proof. Since $P - V = 4$ (Theorem 7), it follows that P and V must either both be odd or both be even. So their sum is even. But $P + V + F = k$, and k is even, therefor F must be even. □

Theorem 9 (Herzog et al. (2014))**.** *Every hole is simply-connected.*

[Not referenced]

Theorem 10 (Herzog et al. (2014))**.** *In a simply-connected polyomino, there is no vertex v that is shared between exactly two cells C and C' such that $C \cap C' = \{v\}$.*

[Not referenced]

Theorem 11. *For a rectilinear polygon, the number of holes H is related to the number of peaks P and valleys V by the following equation:*

$$H = \frac{V - P}{4} + 1.$$

[Not referenced]

Proof. Suppose we have holes $1, 2, 3, \ldots, H$, and the border of holes i has P_i peaks and V_i valleys, and the outer border of our polygon has P_0 peaks and V_0 valleys. The total peaks and valleys is given by:

$$P = \sum_{i=0}^{H} P_i \tag{1}$$

$$V = \sum_{i=0}^{H} V_i \tag{2}$$

And so

$$V - P = \sum_{i=0}^{H} (V_i - P_i)$$

If we consider the border of a hole as a rectilinear figure on its own, we notice that its peaks correspond to valleys in the original polygon, and its valleys corresponds to peaks in the original polygon. So it has V_i peaks and P_i valleys, and so we know by Theorem 7 that $V_i - P_i = 4$, for $i > 0$. For $i = 0$ we have $P_0 - V_0 = 4$, and so we get the following:

$$\begin{aligned} V - P &= V_0 - P_0 + \sum_{i=1}^{H} (V_i - P_i) \\ &= -4 + \sum_{i=1}^{H} 4 \\ &= -4 + 4H \\ &= 4(H - 1), \end{aligned}$$

and so

$$H = \frac{V - P}{4} + 1.$$

□

Theorem 12 (The perimeter criterion). *Suppose we have a figure with peaks of length $e_1, e_2, \ldots$ and a set of tiles with peaks of length $e'_1, e'_2, \ldots$. A necessary condition for the figure to be tileable is that each edge e_i, we have $e_i = \sum_j n_j e'_j$ for some $n_i \geq 0$.*

[Not referenced]

Example 2. *The 9×9 square with the center removed cannot be tiled by 2×2 squares. The area of the figure is 80, and so it satisfies the area criterion. However, we cannot express the edge of length 9 as a multiple of 2, and so it fails the perimeter criterion. In fact, we cannot tile a 9×9 square with a cell removed from anywhere, since we always have at least two borders of odd length. Surprisingly, the minimum number of monominoes necessary to to tile a 9×9 square is actually 17!*

Theorem 13 (The row-criterion). *Suppose our figure rows of length e_i and our tiles have rows of length e'_i. If no rotation is allowed, then a tiling can exist only if $e_i = \sum_j n_j e'_j$ for some $n_i \geq 0$.*

[Not referenced]

Example 3. *The tile in Figure 3 can only tile figures with all rows and columns containing an even number of cells. (In fact, if a row or column is not connected, each piece must have an even number of cells.)*

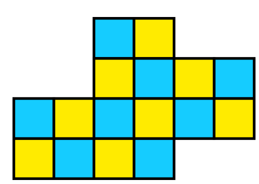


Figure 9:

Dominoes I

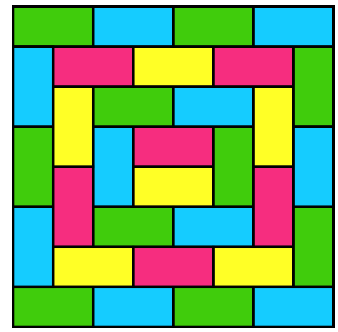


Figure 10: A tiling of a 8×8 square by dominoes.

FIGURE 14 SHOWS some examples of regions that cannot be tiled by dominoes. That some regions cannot be tiled by dominoes is intriguing. How can shapes as small as dominoes not be arranged to fit into a region as big as in Figure 14(e), when there are so many options?

This is the topic of the first section: understanding why some regions have a tiling by dominoes and others don't.

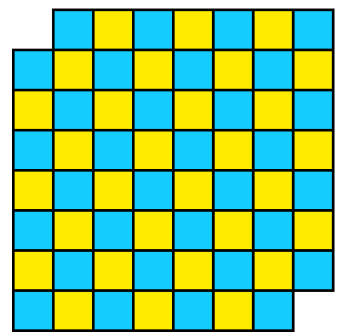


Figure 11: The mutilated chessboard.

Figure 47 provides us with another mystery. In each region, there are dominoes that fit into the tiling in just one way (they are marked in yellow). In the first region it is easy to see why this must be so, but what is happening in the last region? How is it not possible to find a tiling so that at least some of those dominoes lie in a different position?

This is the topic of the second section: understanding how the same region can be tiled in different ways and how the region can force dominoes into certain positions.

A FAMOUS PROBLEM, the *mutilated chessboard problem*[6], illustrates some key ideas from each section.

[6] The mutilated chessboard problem, was first proposed by Max Black in Black (1947), and has been discussed in various places, including (Golomb, 1996, p. 4), (Martin, 1991, p. 1-4, 7-9), (Mendelsohn, 2004) and Engel (1998).

Consider an 8×8 chessboard, with two opposite corners removed as in Figure 11. Can the board be tiled? If you know this problem, you know that it cannot. Because, every domino must cover exactly one black and one white square, and with opposite corners removed, there are more cells of one color than the other, and so a tiling is impossible. Of course, this principle can be applied to any region, and gives us a valuable tiling criterion. We will use this as the basis for constructing tiling criteria more powerful than the area criterion (Theorem 1) that we gave in the introduction.

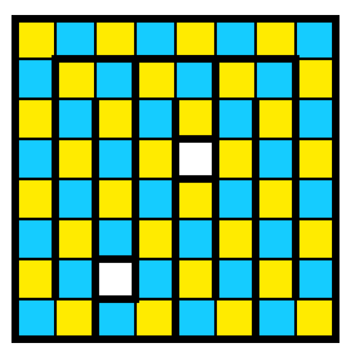


Figure 12: Dividing the chessboard into two strips

If we remove two squares of opposite colors instead, is a tiling always possible? The answer is, *yes*, a tiling is always possible. Here they key is to notice that we can divide the board into two strips, as shown in Figure 12. And because the cells have different colors, no matter where we remove the two cells from, the end-points of each

strip must have different colors, and so it has an even number of cells. Each of these strips has an even number of cells, and is tileable in the obvious way. (We will go over this logic in more detail once we made some proper definitions.)

One way to show a region is tileable, is to partition it into strips with an even number of cells. If the end point of a strip of more than two cells is next to its starting point, we say the strip is closed. Closed strips have at least two tilings, and this is the basis of the second section. We will see that cells that can only be tiled one way can never be part of a closed strip, and that all tilings of a region can be obtained from the two tilings of each closed strip in it.

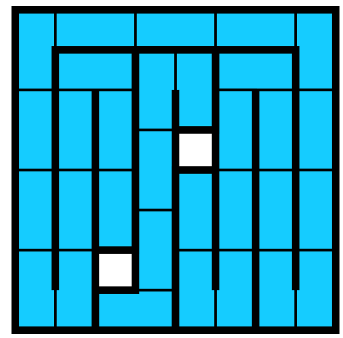


Figure 13: A tiling along the strips.

Tiling Criteria

In this section we develop some tools with which we can tell whether a region is tilable by dominoes or not.

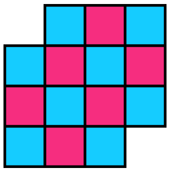
We could, of course, use brute force to check all the regions in Figure 14 (or get a computer to do it). There are two reasons to look for something better:

- So that we can write faster programs. While a naive program might deal with all our examples in seconds, scientists using tilings to understand how molecules fit in solids need something better. Their "tiling problems" may be regions with millions of cells!
- So that we can understand how these tilings work, and understand other related phenomena. Section two gives a taste of this.

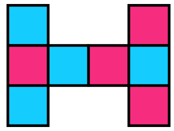
Our goal is to develop four techniques:

- A flow criterion: a technique of coloring cells in a region and analyzing the counts of dominoes that must cross the borders of subregions.
- Cylinder reduction: a geometric transformation that preserves the tileability of a region and can be used to simplify regions.
- The marriage condition: another criterion for determining whether a tiling exists.
- A generalized way of coloring regions to exhibit their untiliablity.

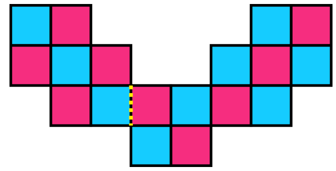
We do not arrive at the state-of-the-art in solving domino tiling problems (just yet), however, the tools will help us deal with all the



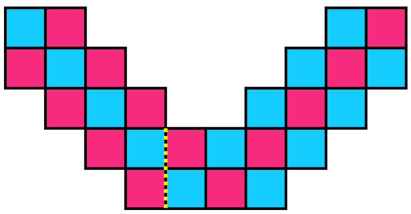
(a) An example of an unbalanced polyomino.



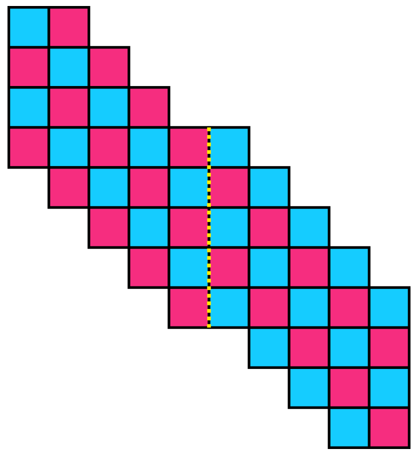
(b) An example of a balanced non-compact polyomino that cannot be tiled.



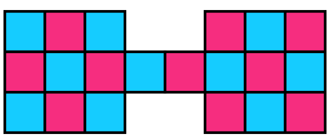
(c) An example of a balanced compact shape that cannot be tiled.



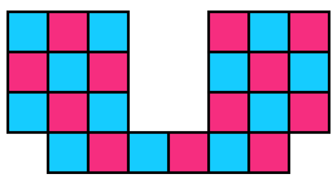
(d)



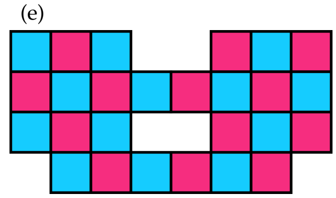
(e)



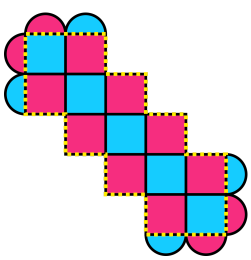
(f)



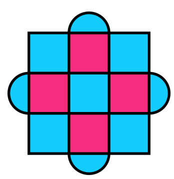
(g)



(h)



(i) No matter how this region and the next one are extended at the blobs, the resulting regions will not be tileable



(j)

Figure 14: Regions that cannot be tiled with dominoes.

regions in our examples in Figure 14, and give us an intuitive understanding of how to tackle tiling problems.

A key point to understand from this section is: the border of a region is important, and tell us a lot about how a region can be tiled.

Flow

SINCE ALL TILINGS MUST SATISFY the area criterion (Theorem 1), we know a tiling by dominoes can exist only if the area of the region we wish to tile is even. Not all regions with even area are tileable (all the regions in Figure 14 have even area).

Now suppose we divide a region into two parts, each with an odd area. If the original is tileable, then we know that there must be at least one domino that lies in both of the two parts. In fact, we know the number of dominoes that crosses the border between the two parts must be odd, otherwise the remaining regions will have odd area and not be tileable. Figure 15 illustrates this principle.

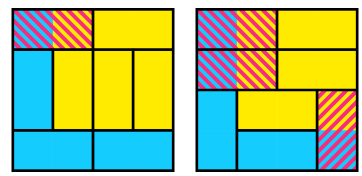


Figure 15: Two examples of the tiling of a square. The square is divided into two partitions (yellow and blue). Since both partitions have odd area, any tiling such as the two shown above must have an odd number of dominoes that cross the border.

Theorem 14 (Border Crossings Theorem)**.** *In a subregion S of a region with a tiling, the number of dominoes that cross the border of the subregion must have the same parity*[7] *as the area of the subregion.*

[7] Even or odd.

[Referenced on pages 15 and 31]

Proof. Let k be the number of dominoes that cross the border of S. Each of these dominoes has only one cell in S. If we remove these cells to form a new region S', we have a region that is tileable, with area $|S'| = |S| - k$. Since S' is tileable, $|S'|$ must be even (by Theorem 1), and so $|S|$ and k must either both be odd, or both be even. □

Below we give two applications of this theorem; as a theorem and an example.

A *bridge* is a sequence of cells in a region such that each has exactly two neighbors, and such that removing any cell in the bridge will leave the region disconnected. For example, Figure 14(f) and (g) each has a bridge, Figure 14(h) does not (since no cell splits the region when removed). Figure 14(c) also does not have a bridge; although there are single cells that will split the region if removed, they have more than two neighbors.

Theorem 15. *Each cell in a bridge can be tiled in only one way.* [8]

[8] Such cells are called *frozen*. We will define this concept in the next section.

[Not referenced]

Proof. Let R be a region with a bridge, and let v be a cell in that bridge. Because v is a cell in a bridge, it has exactly two neighbors– let's call them u and w. Further, $R - \{v\}$ is disconnected. Let S_u and S_w be the two disconnected subregions that contains u and w respectively. The border of each of these share exactly one edge with the cell v, in particular, S_u shares a border with v at the edge between u and v.

Now consider S_u. If $|S_u|$ is odd, then the number of dominoes that crosses the border of S_u is odd. The only place where a domino can cross is for a domino to cover both u and v, and therefor any tiling of R must have a domino in this position.

If on the other hand $S|S_u|$ is even, then the number of dominoes that crosses the border of S_u is even. Since there is only one place where a domino can cross, it means the number of dominoes that cross must be 0. Therefor, a single domino cannot cover both u and v, and so, a single domino must cover v and w. Therefor, every tiling of R must have a domino in this position.

Taken these together, v can only be tiled one way. The same argument applies to all oither cells in the bridge, and therefor the bridge has a unique tiling in R. □

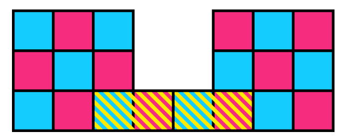


Figure 16: The cells marked yellow can be tiled in only one way.

The next example uses the area principle in a more sophisticated way to prove that a tiling does not exist (Mendelsohn, 2004). [9]

[9] The author mentions this idea has been known before.

Example 4. *Let's look at the mutilated chessboard. The top row has an odd area. Therefore, an odd number of dominoes must cross its border, and only vertical dominoes that also lie in the second row can do that (otherwise, parts of dominoes will fall outside the mutilated chessboard).*

The second row has even area, and so an even number of dominoes must cross its border. We already have an odd number of dominoes that cross the border from the top, therefore we must have an odd number of vertical dominoes that cross the border to the bottom.

Following the same argument, we get that there must lie an odd number of vertical dominoes between each pair of adjacent rows. There are 7 such pairs, so we can conclude the total number of vertical dominoes must be odd.

Applying this idea to the columns, we find there must be an odd number of horizontal dominoes too. So the total number of dominoes must be even. But to cover the 62 squares we need 31 dominoes, which is odd. And therefore, a tiling is impossible.

This example shows that we can always determine the parity of the number of vertical and horizontal dominoes in a tiling. For a tiling to exist, this must be consistent with the parity of the total number of dominoes (which we can deduce from the area).

SATISFYING THE AREA CRITERION is not enough (all the regions in Figure 14(a)-(h) are untileable and have even area), and checking the parity as in the previous example is also not enough (for example, Figure 14(f)).

We will now look at the color argument we discussed at the beginning of the chapter, and see how far it gets us. The **checkerboard coloring** plays an important role in our discussions going forward, and to make it easier to talk about and prove details we introduce some additional notation and terminology.

In a region R with checkerboard coloring applied, let $\mathcal{W}(R)$ denote the white cells in R, and let $\mathcal{B}(R)$ denote the black cells in R.The **deficiency** of R is defined as

$$\Delta(R) = |\mathcal{B}(R)| - |\mathcal{W}(R)|. \tag{3}$$

If the deficiency of a region is 0, the region is **balanced**.[10]

[10] The term *balanced* is used informally in (Golomb, 1966, p. 17). The term , as well as the white and black functions, are defined explicitly in for example Thiant (2003).

The functions $\mathcal{W}$, $\mathcal{B}$ and Δ depend on which of two ways the image has been colored. However, the absolute value of the deficiency and being balanced are inherent features of a region, and are independent of the coloring used.

Theorem 16 (Checkerboard Criterion)**.** *For a region to be tileable by dominoes, it must be balanced (Golomb, 1996, p. 4).*

[Referenced on pages 20 and 24]

Using this criterion, we can prove Figure 14(a) is not tileable. However, it *still* is not enough, since all the regions in Figure 14(b)-(h) are balanced and untileable.

Problem 2. *Find some examples of non-tileable balanced regions.*

What is the largest the deficiency can be? Figure 17 shows we can make the deficiency as large as we want by extending it as shown. However, it is bounded by the number of cells, as stated in the following theorem.

Theorem 17. *The deficiency of a connected region R is bounded by the number of cells as follows:*

$$|\Delta(R)| \leq \frac{|R|+1}{2}.$$

[Not referenced]

Proof. Let $B = \mathcal{B}(R)$ and $W = \mathcal{W}(R)$. WLG, assume that $B > W$, so $\Delta(R) > 0$. We will shows that $B \leq 3W + 1$, from which the result follows. Because if $B \leq 3W + 1$, we have $2B - 2W \leq W + B + 1$, that is, $2\Delta(R) \leq |R| + 1$, or $\Delta(R) \leq \frac{|R|+1}{2}$.

We now prove $B \leq 3W + 1$. Suppose not; that is, suppose $B > 3W + 1$. Then there must be at least one white cell with 4 black neighbors, with none of these 4 black cells having any other white neighbors. If there are no other cells than these 5 cells, we have $4 = B = 3W + 1$, so there must be more. But then these 5 cells cannot be connected to any other cells, and we have a disconnected region which contradicts our hypothesis that the region is connected. □

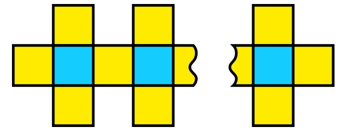


Figure 17: A family of regions that obtain maximum deficiency.

This upper bound is achievable by regions like the one shown in Figure 17.

In the same way that we turned the area criterion into something more useful by looking at what must happen at the border of subregions, we now turn the color criterion into something more powerful by looking at the border of subregions.

Let R be a region with the checkerboard coloring applied, and let S be a subregion of R. If w is the number of dominoes covering the border of S with a white cell inside S, and b is the number of dominoes covering the border with their black cells inside R, then we define the **flow**[11] of S as

$$\phi(S) = b - w. \tag{4}$$

[11] This definition is essentially given in Saldanha et al. (1995).

See Figure 18.

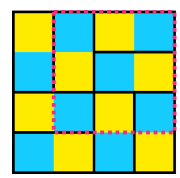


Figure 18: Let S be the region inside the red border. Then $b = 2$, $w = 1$, and so $\phi(S) = 2 - 1 = 1$. Also, $\mathcal{B}(S) = 5$, $\mathcal{W}(S) = 4$, and so $\Delta(S) = 5 - 4 = 1$.

When $S = R$, the flow is 0, since there are no dominoes that cross the border, and so $w = b = 0$.

Theorem 18 (The Flow Theorem)**.** *Let S be a subregion of R, and suppose R has a tiling by dominoes. Then the flow of S equals the deficiency of S, that is,*

$$\phi(S) = \Delta(S). \tag{5}$$

[Referenced on pages 16 and 31]

Proof. Let W (B) be the number of white cells inside S, let w (b) be the number of dominoes that cross the border with their white (black) cells inside S, and let k be the number of dominoes completely in S. Then $B - W = (b + k) - (w + k)$, and so $B - W = b - w$, and thus $\Delta(S) = B - W = b - w = \phi(S)$. □

Note the similarities between border crossings theorem 14 and the flow theorem. Both give us information about what happens at the border based on what is inside the region. But the flow theorem gives us much more information; Theorem 14 merely tells us the parity

of the number of dominoes that cross the border. The flow theorem gives us the difference of the two different types of dominoes.

Also note that the area must have the same parity as the flow: if the area is even, then W and B are both odd or both even. In either case, their difference is even. This means with the flow theorem in place, the border crossing theorem is now redundant. [12]

[12] One reason to differentiate the two theorems is that the border crossings theorem is easier to generalize to other tile sets.

The following examples show how to use the flow theorem to prove a region is not tileable.

Example 5. *Consider Figure 14(c), and let's choose the subregion as the shape left of the dotted line. Now since* $W = 3$ *and* $B = 4$*, we have* $|W - B| = 1$*, so we know a domino must cross the dotted line. Removing this domino partitions the shape into two shapes, each of which is untileable because they don't satisfy the area criterion.*

Example 6. *Consider Figure 14(d), and choose the subregion as the region the right of the dotted line. We have* $|B - W| = 2$*, which implies dominoes must overlap the dotted line in two places. But this means* $b = w = 1$*, and so* $|b - w| = 0$*, which is impossible if the region is tileable. Therefore, it is not tileable.*

Example 7. *Consider Figure 14(e). Partition it in halves by a vertical cut through the middle. Then* $|B - W| = 4$*, but the maximum value that* $|b - w|$ *can have is 3. Therefore, the region is not tileable.*

Example 8. *Consider Figure 14(i), and consider the colored subregion. The deficiency* $|B - W| = |8 - 5| = 3$*. This means, at least three dominoes must overlap the border. However this is done, we are always left with a region with 4 black and 3 white squares, which is untileable. Therefore, the entire region is untileable.*

If you tried your hand at Problem 2 and followed the examples above, you may have noticed that you can create an untileable region by having an abundance of black on the one side of the region, and a abundance of white on the other side, with a choke point between the two parts. The following theorem gives a formulation of this idea.

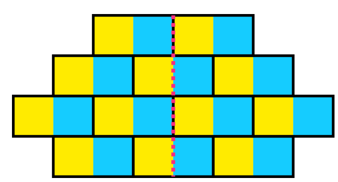


Figure 19: The deficiency of the left partition is -2, so the length of a cut through the center must be at least 3.

Theorem 19. [13] *Suppose we apply the checkerboard coloring to a tileable region, and partition it into two subregions with a straight cut. If one subregion has* W *white cells and* B *black cells, then the cut must have length at least* $2|B - W| - 1$*.*

[13] In Kenyon (2000) the author mentions that a very similar theorem has been proven in Fournier (1996) (in French).

[Not referenced]

Proof. Assume $B \geq W$. From the flow theorem (Theorem 18), we know that the number of tiles that cross the cut must be at least $B - W = b - w$. From this we get $b = B - W + w$, so $b \geq B - W$ (since w is non-negative). So the number of dominoes that cross the cut with

a white cell inside the subregion must be at least $B - W$. Along a straight cut, there are at most $\left\lceil \frac{L}{2} \right\rceil$ places where this can happen, so it needs to have length at least $2(B - W) - 1$.

If we assume $W \geq B$, we can show the length of the cut must be at least $2(W - B) - 1$ following the same argument as above, reversing the roles of black and white.

Putting these together, we arrive at the result: the length of the cut is at least $2|W - B| - 1$. □

See Figure 19 for an example.

Problem 3. *What if the cut is not straight?*

You may also have noticed that to create unbalanced regions or parts of regions, you have to manipulate the border of the region so you have a lot of corners of the same color. Also, we have already seen some theorems that relate the border of a region to what is going on inside. The following theorem shows we can determine the deficiency from what is going on at the border alone.

Theorem 20. *Let b be the number of black edges on the border, and w be the number of white edges on the border. Let B be the number of black squares and W be the number of white squares. Then $b - w = 4(B - W)$.*

[Referenced on page 18]

Proof. Consider building a region cell by cell. At each stage, we can either add a white or a black square. If we add a white square, it is a neighbor of 0, 1, 2, 3, or 4 other cells, all of which must be black. Each exposed edge must be white, and each unexposed edge must reduce the total number of black edges. So the total amount that $b - w$ decreases is 4. A similar argument shows that if we add a black square, $b - w$ is increased by four. In other words: $b - w = 4(B - W)$. □

It follows that $B = W$ if and only if $b = w$.

Problem 4. *A corner cell is a cell with one or two neighbors. Let R be a region such that each of its corner cells have no neighbors that are on the border (that is, all neighbors of each corner cell are interior cells). Prove R is untilable.*

Theorem 21 (Csizmadia et al. (1999))**.** *If all the edges if a simply connected region have odd length, the region is unbalanced.*

[Referenced on page 18]

Proof. [14] Suppose one corner edge is black. Then all corner edges are black. This means the ends of sides are all black, and so for each side i, we have:

$$B_i - W_i = 1,$$

and summing over all sides:

$$\sum_{i=1}^{n}(B_i - W_i) = n.$$

But $\Delta(R) = \frac{\sum_{i=1}^{n}(B_i - W_i)}{4}$ (by Theorem 20), and thus $\Delta(R) = \frac{n}{4} > 0$, and so R is unbalanced. □

[14] The proof here is new. The proof in Csizmadia et al. (1999) is quite complicated.

The theorem does not hold for regions with holes. For example, the 3×3 square with its center removed has all its sides odd, but it is balanced and even tileable.

Theorem 22. *A balanced polyomino must have at least two sides of even length.*

[Not referenced]

Proof. For the polyomino to be balanced, it must at least have one even side (Theorem 21). We also know the perimeter must be even (Theorem 4). Therefor, the number of odd sides must be even. But the total number of sides must be even (Theorem 5), and so the number of even sides must *also* be even. Thus, there must be two or more even sides. □

The Marriage Theorem

INFORMALLY, THE *marriage theorem* states the following[15]: Suppose we have a group of k white cells from a region R, and they have n black neighbors in R. If $n < k$, then no tiling exists. Moreover, if every group of white cells in R has at least as many neighbors as white cells in the group, a tiling exists.

[15] This is a specific version of a much more general theorem given in Hall (1935) that is now called the *marriage theorem*, with applications that go beyond tiling. Covering the more general results fall outside the scope of this essay, but we give some references in the *Further Reading* section. This version is given in Ardila and Stanley (2010).

Before we prove it, we need some terminology to make it easier to make our statements.

A subregion S of R is called a **white patch** of R if all the neighbors of its white cells are also in S, and there are no other black cells in S. In other words, all black cells in S have at least one neighbor also in S. A similar definition can be made for **black patch**.

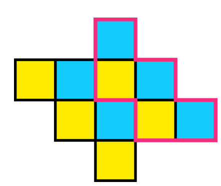


Figure 20: In this region, the cells surrounded by a pink line is a black bad patch. The rest of the region is a white bad patch.

A region itself is a patch. If a patch is not equal to the region we call it a **proper patch**. A white patch can be, but is not necessarily, a black patch. In fact, if R is connected, a white patch is only also a black patch if it is the entire region.

Theorem 23. *Suppose R is connected, and S is both a white patch and a black patch. Then $S = R$.*

[Not referenced]

Proof. Suppose R has some cells that are not in S. Since R is connected, we must have at least one of these cells be a neighbor to a cell in S. Let's call this cell u, and its neighbor in S, v. Suppose v is black, then because S is a black patch, all its neighbors must lie in S, and this contradicts that u lies outside S. And if v was white, because it is a white patch, all its neighbors must lie in S. Therefore, there can be no cells in R that are not also in S, and therefore $S = R$. □

Theorem 24. *Let S be a white patch of a region R. Then $R - S$ is a black patch of R. Similarly, if S is a black patch of R, then $R - S$ is a white patch.*

[Referenced on page 19]

Proof. We only prove the first part; the second part can be proven with the exact same argument with the roles of white and black reversed.

Suppose S is a white patch, but $R - S$ is not a black patch.Then there is a black cell u in $R - S$ with a neighbor v *not* in $R - S$. Then v must lie in S, and it is white. Since S is a white patch, all the neighbors of v, including u must lie in S. We arrive at a contradiction, and so $R - S$ must be a black patch. □

We call a white patch **bad** if it has fewer black than white neighbors, and **good** otherwise. Similarly for black patches.

Theorem 25. *A balanced region has a bad black patch if and only it has a white bad patch.*

[Referenced on page 20]

We only prove the first part, the second part follows from the same argument reversing the roles of white and black.

Proof. If.

Suppose that S is a bad white patch. Then $R - S$ is a black patch (Theorem 24). Since S has fewer black than white cells and R is balanced, $R - S$ must have fewer white cells than black cells, and so it is bad. □

It follows directly from this that if all white patches in a region are good, then so are all black patches.

Theorem 26 (The Marriage Theorem)**.** *A region is tileable if and only if it has no bad patches.*

[Referenced on pages 21, 22 and 31]

Proof. If.[16]

Suppose all the white patches of a region are good. It follows that all the black patches of the region must be good too (and vice versa) by Theorem 25. We can now partition the region into two subregions S and $R - S$ in one of two ways:

(1) If all the proper white patches of R are unbalanced, then each patch of R must have strictly more black cells than white cells. Let S be a white cell and its neighbor. Then S is tileable (with a single domino), and $R - S$ is a region with all its white patches good. This follows from the fact that we removed only one black cell, so the number of black cells in each patch can can drop by at most 1, and since these are strictly bigger than the number of white cells, after the drop there must be at least as many black cells as white cells in the patch.

(2) If at least one of the patches in R is balanced, we let S be such a patch. Then S must have all its white patches good, and $R - S$ must have all its black patches good (and so, it must also have all its white patches good). Both partitions must also be balanced.

    Note that a patch in S or $R - S$ need not be a patch in R.

We can continue this process, and it must eventually end, since the number of cells in R is finite. And so, we eventually arrive at a bunch of subsets of two cells each, all tileable by a single domino, and so the entire region must be tileable by dominoes.

Only if. If the region is not balanced, we know it is not tileable by Theorem 16.

Suppose then it is balanced, and suppose it has a bad white patch. (If it had a bad black patch, it must also have a bad white patch by Theorem 25, so there is no loss in generality.)

If there is a tiling, each white cell in the white patch has an associated black neighbor that lies in the same domino, and there must be at least as many of these black cells as white cells. However, the patch is bad so this is not the case, a contradiction. Therefore no tiling exist. □

[16] The logic of this part of the proof follows the proof in (Kung et al., 2009, p. 56), due to Easterfield (Easterfield, 1946) and Halmos and Vaughan (Halmos and Vaughan, 2009).

We finally have a criterion that can work for all tilings. However, the problem is that it can be difficult to find a bad patch, or to show there aren't any. For example, in a region R, the flow theorem is easier to apply (although, of course, you *could* find a bad patch.)

That does not mean it is not useful: We will use this theorem a few times to prove some other things about tilings, and it also gives us a

way the come up with untileable regions easily as the next theorems show.

An **extension** of a region R is a region P such that R is a subregion of P. Informally, an extension of R is some region formed by adding cells to R.

Theorem 27. *Suppose R is a region with an odd number of cells. If we apply the checkerboard coloring such that $\mathcal{W}(R) < \mathcal{B}(R)$, then any extension of R that adds no new neighbors to black cells in the polyomino is untileable.*

[Not referenced]

Proof. Let the original region be R, and its extension P. Since $W < B$, R is a bad patch with respect to itself. But since we do not add any neighbors to form P, R is also a bad patch with respect to P, and therefor untileable. □

Theorem 28. *If a region has a bad patch, it has a connected bad patch.*

[Not referenced]

Proof. Let S be the bad patch. WLG assume that $\mathcal{B}(S) < \mathcal{W}(S)$. Now partition S into connected disjoint sets S_i such that the (black) neighbors of all the white cells in S_i is also in S_i. We want to show one of these sets is a bad patch. If each S_i is a good patch, then $\mathcal{B}(S_i) \geq \mathcal{W}(S_i)$ for all i. But then $B(s) = \sum_i \mathcal{B}(S_i) \geq \sum_i \mathcal{W}(S_i) = \mathcal{W}(S)$, a contradiction. □

This theorem shows the process of generating untileable regions described above can generate every untileable region.

Cylinder Deletion

We can also use the Marriage Theorem (Theorem 26) to prove certain reductions do not affect the tileability of a region.

A **vertical n-cylinder** is a simply connected region where each row has n cells (see Figure 21). A **horizontal n-cylinder** has n cells in each column (Hochberg, 2015).[17]

[17] Cylinders also play an important role in tiling extensions and tilings of the infinite strip.

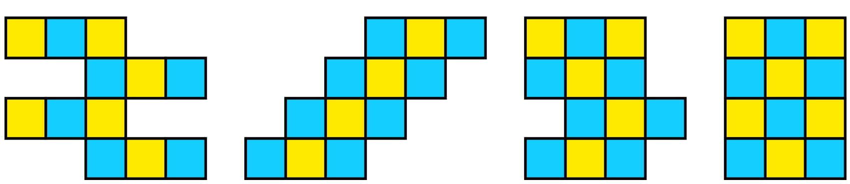


Figure 21: Examples of vertical 3-cylinders.

An n-cylinder is tileable by dominoes if n is even, since each row has an obvious tiling by horizontally-placed dominoes.

Suppose a vertical cylinder S is a subregion of R such that it shares is top and bottom borders with the border of R. If we remove S from R, and move the two pieces together, we get the new region $R \ominus S$. We call this operation a **deletion**. An analogous definition can be made for a horizontal cylinder S. After a deletion, we my be left with a region that has a barrier. For example, in Figure 22 we delete a cylinder from the double-T region. This yields a new shape with two internal barriers that cannot be crossed by dominoes. In particular, it means the two top cells are not neighbors of each other. Note that we cannot delete a horizontal cylinder from the reduced region because of the barriers.[18]

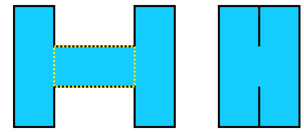


Figure 22: Cylinder deletion. (a) The cylinder S is the shape contained in the yellow dotted line, and R is the entire region. (b) $R \ominus S$. Note the barriers that cannot be crossed by dominoes.

[18] We will later see how this geometrical operation is equivalent to simplifying the border word algebraically.
The operation is also very similar to deletions on rhombus tilings, also called *contractions*. In this context, the deleted shape is called a *de Bruijn section*. See for example Chavanon and Remila (2006) and Chavanon et al. (2003).

Theorem 29. *Let S be an n-cylinder with n even. Then R is tileable, if $R \ominus S$ is tileable.*

[Referenced on page 27]

Proof. WLG, assume S is a vertical cylinder.

Suppose $R \ominus S$ is tileable. Then we can find a tiling for R as follows: In $R \ominus S$, the cut line is either covered by a domino or not. Tile all the cells in R that are also in $R \ominus S$ the same way. If there are dominoes that cross the cut in $R \ominus S$, there will be two dominoes that cross the two cuts in R (in the same row). So any row in S will either have n untiled cells, or $n-2$ untiled cells. Since this number is even, we can fill the row with horizontal dominoes. this gives us a tiling for R.

□

Unfortunately, the converse of this theorem is not true (see for example Figure 23). However, a somewhat weaker version of it is true. A deletion is called **safe** when all the cells in $R \ominus S$ have neighbors in the same directions as they had in R.

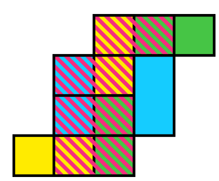


Figure 23: An example of a region R that is tileable, but $R \ominus S$ is not, for the 2-cylinder S marked pink.

Theorem 30. *Suppose S is a n-cylinder of R with n even, and that deleting it from R is safe. Then if R is tileable, then so is $S \ominus R$.*[19]

[19] I suspect something stronger is true, but have not been able to work out the exact details.

[Not referenced]

Proof. The reduced region $R \ominus S$ has exactly the same neighbor setup as R. If R is tileable, then by the marriage theorem (Theorem 26), each set S of white cells has at least $|S|$ black neighbors. This must also hold for $R \ominus S$, since if it had a bad patch, so would R. Therefore, $R \ominus S$ is tileable. □

This method gives us a way to reduce some regions to a more manageable level.

But this method *also* gives us a way to construct a tiling for R if we can find a tiling for $R \ominus S$.

Example 9. *In Figure 24 we show how a tiling can be constructed for a given region.*

First (shown in the left column of Figure 24), we successively delete 2-cylinders from the region, until the resulting region is simple enough to tile. We then tile it (if we couldn't, then its possible that no tiling exists for R. We do not know for sure, unless all the removals were safe).

Then, working backwards (shown on the right), we reinsert cylinders until we have rebuilt the original region. If the cutline goes between dominoes, we simply insert a domino perpendicular to the cutline; otherwise we insert a domino domino perpendicular to the cutline offset by a cell.

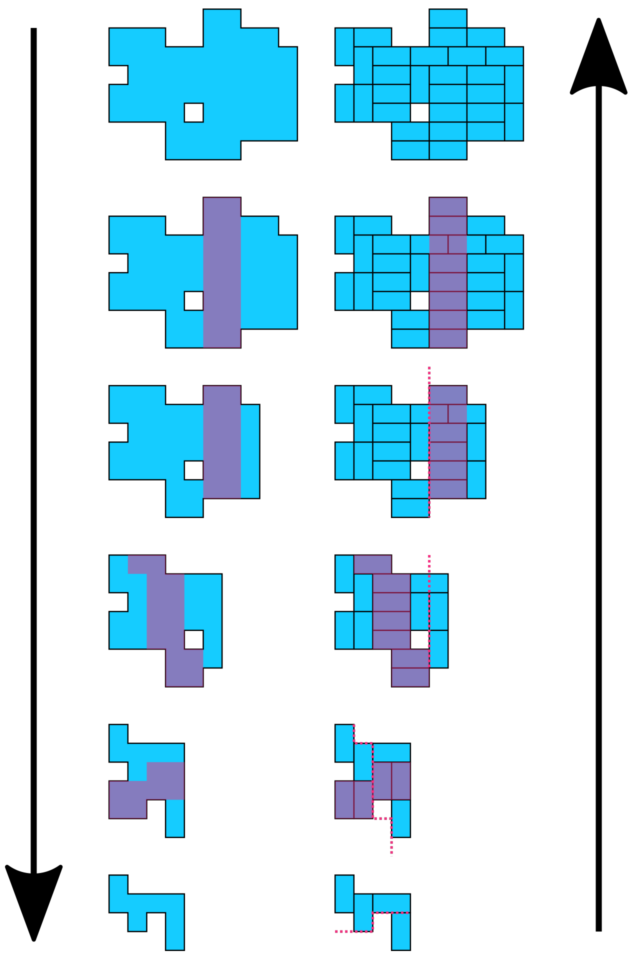


Figure 24: How to use cylinder deletion to find a tiling. On the left, from top to bottom, we delete 2-cylinders until we cannot. This region is easy to tile, shown on the bottom right. We then reinsert 2-cylinders in the reverse order (shown on the right-hand side from bottom to top). At each stage we complete the tiling in the obvious way.

Problem 5. *Establish the tileability of the regions in 14 by deleting cylinders until each region is manageable.*

Problem 6. *Give some examples of regions that do not allow us to delete a cylinder from them. Are any of them tileable?*

Problem 7. *Can you characterize the regions from which we can delete cylinders?*

A **bar graph** is a polyomino with columns all starting on the same horizontal line Bousquet-Mélou et al. (1999).[20] A bar graph is uniquely identified by a vector $[a_1, a_2, \ldots a_n]$, where a_i is the number of cells in column i.

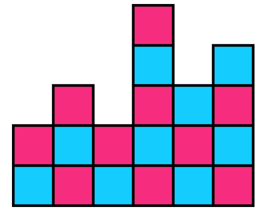


Figure 25: A bar graph corresponding to the vector $[2, 3, 2, 5, 3, 4]$.

A **Young diagram** is a bar graph with columns in non-increasing order (See for example Pak, 2000).[21] In a young diagram, we have $a_i \geq a_j$ when $i < j$.

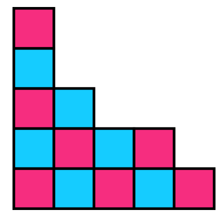


Figure 26: Young diagram corresponding to the vector $[5, 3, 2, 2, 1]$.

Theorem 31. *A balanced Young diagram must either have (at least) two adjacent columns equal, or (at least) two adjacent rows equal.*

[Referenced on page 24]

Proof. If a Young diagram does not satisfy those conditions, it must be a polyomino of the form $[n, n-1, n-2, \ldots, 2, 1]$. If we apply a checkerboard coloring such that the cell in the right most column is black, starting from the right, every second column has one more black square than the one on the right, and so $B - W = \lceil \frac{n}{2} \rceil$, and hence the polyomino is not balanced. Therefore, a balanced polyomino cannot have this form, and so a balanced polyomino must satisfy the conditions given. □

Theorem 32. *A Young diagram is tileable if and only if it is balanced.*

[Referenced on page 24]

Proof. If. By Theorem 31 we know the polyomino has either two adjacent columns of equal length, or two adjacent rows of equal length. In either case, we can delete a 2-cylinder from the region. Since the resulting region must still be balanced, the conditions apply again, and we can repeat the process. The process must eventually end with the empty region, and this proves the Young diagram is tileable.

Only if. If a Young diagram is tileable it is balanced by Theorem 16.

□

We already saw a class of polyominoes for which the Checkerboard Criterion (Theorem 16) is enough (namely Young diagrams,

[20] Also called *Manhatten polyomino* (See for example Bodini and Lumbroso, 2009)

[21] Also called a *Ferrers diagram* (See for example Delest and Fedou, 1993) or *trapezoidal polyomino* (See for example Bodini and Lumbroso, 2009) or *partition polyomino* (See for example **?**).

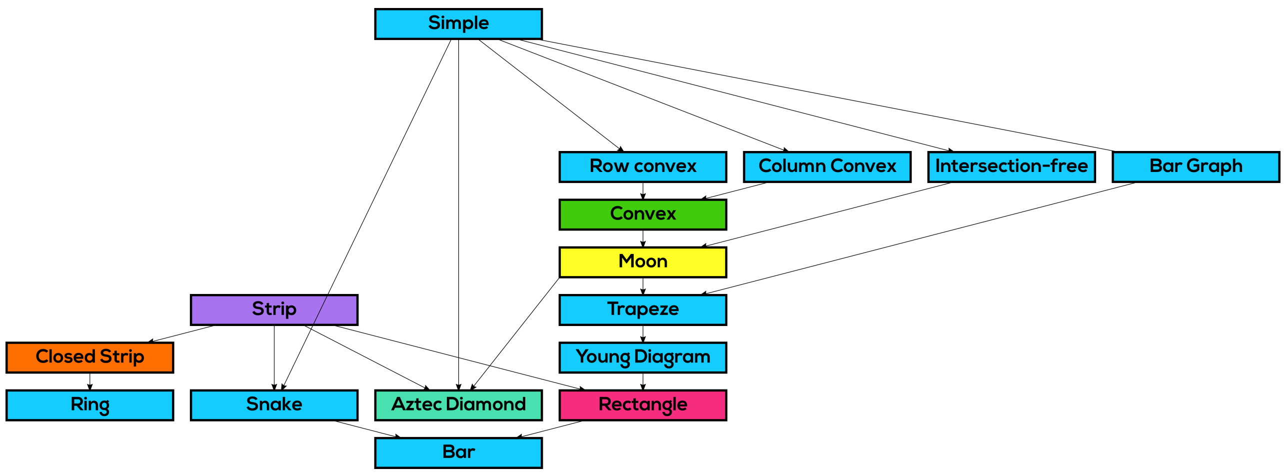


Figure 27: The relationship between some polyomino classes.

see Theorem 32), and we may wonder if there are other classes of polyominoes this is true.

There is, and this section we prove that a large class of polyominoes—that include Young diagrams—are tileable when they satisfy the checkerboard criterion. I will give three proofs of this fact, but before we get there we need a few definitions and helper theorems.

A domino in a tiling of some region is called **exposed** if at least one of its long sides lies completely on the border of the tiled region.

Theorem 33 (Neretin (2017))**.** *Any domino tiling with more than one tile has at least two exposed dominoes.*

[Referenced on pages 26, 27 and 42]

Proof. We give an algorithm to produce two exposed long edges.

(1) Grab any tile. If it is horizontal, look at its top (long) side. Otherwise look at the right (long) side.

(2) Introduce the coordinates: $(x, y) = (0, 0)$ at the middle of that side.

(3) If that side is free, we're done. If it is not, there must be another tile blocking it (maybe partially). Switch to that tile (or the rightmost/topmost of the two, if there are two).

(4) If it is horizontal, look at its top side. Otherwise look at the right side.

(5) Check the value of $x + y$ at the middle of that side. Make sure it increases when we step from a horizontal tile to vertical or vice versa, or (in the worst case) stays constant when we step to a tile of the same orientation.

(6) Go to step 3 and continue. It must end somewhere, for there are only so many tiles and they never repeat. (There can't be a cycle of tiles in different orientation, because $x + y$ increases when we change orientation, and never decreases. Neither can there be a cycle made of horizontal tiles only, for in that case y steadily increases on every step.)

To locate the second exposed long side, return to the initial tile and repeat everything in the opposite direction.

In fact, there must be at least two *opposite* long edges exposed.

The procedure to find the first edge always produces a right or top edge, and the procedure to find the second edge always produces a left or bottom edge. If the edges are opposite, we are done. Suppose they are not different, and WLG let the first edge be a right edge and the second edge a bottom edge. Use the procedure to find a third edge, this time going top left (rotate the coordinate system 90 degrees anticlockwise). The procedure must either produce a left or top edge; in either case, it is opposite with one of the other two exposed edges. □

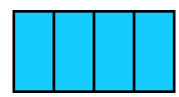


Figure 28: An example of a tiling that achieves the minimum number of exposed edges.

There are tilings that achieve the minimum number of exposed edges. An example is shown in Figure 28.

It is easy to modify this proof to apply to all rectangles (in the case of a square, all edges are "long").

Problem 8. *Extend Theorem 33 to all rectangles. Can it be extended to an even bigger class of regions?*

Rows are **comparable** if the column coordinates of one is a subset of the others.

A polyomino in a fixed orientation is **row convex** when in each row, there are no spaces between any two cells.

A **stack polyomino** is a bar graph that is row-convex and each pair of rows is comparable.[22]

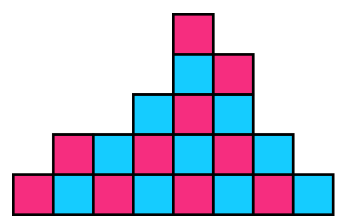


Figure 29: A stack polyomino corresponding to the vector $[1, 2, 2, 3, 5, 4, 2, 1]$.

Theorem 34. *A stack polyomino contains a cylinder that can be deleted.*

[Referenced on page 27]

Proof. By the domino exposure theorem (Theorem 33, we have two opposite long edges exposed.

(1) If these are top and bottom edges, then we have at least two columns equal; these column form a cylinder that can be deleted.

(2) If these are left and right edges, we can delete a horizontal cylinder that lies between these two edges. This is always possible, since the polyomino is column convex.

[22] Also called a *trapeze* Beauquier et al. (1995).

□

Theorem 35. *Suppose R is a stack polyomino, and S is a cylinder that can be deleted from R. Then $R \ominus S$ is a stack polyomino.*

[Referenced on page 27]

Proof. Suppose the polyomino is $[a_1, \ldots, a_m, \ldots, a_n]$, and $a_{m-1} < a_m \geq a_{m+1}$.

Suppose the cylinder we delete is vertical. This is equivalent to removing two adjacent columns; the resulting polyomino is a bar graph, whose columns still satisfy thee inequalities that make the polyomino a stack polyomino.

Suppose the cylinder we delete is horizontal, and it lies in columns k to k'. Then, $a_{k-1} \leq a_k + 2$, and $a_{k'} + 2 \geq a_{k'+1}$. The new polyomino is a bar graph with $[a_1, \ldots, a_k - 2, \ldots, a_m - 2, \ldots, a_{k'2}, \ldots, a_n]$. Combining all the inequalities, we see this new bar graph is a stack polyomino. □

A **jig-saw** region is a balanced stack polyomino with all cells of one color removed from the top row.

Theorem 36 (Bougé and Cosnard (1992))**.** *A balanced jig-saw region is tileable by dominoes.*

[Referenced on page 28]

Proof. Suppose R is a balanced jig-saw region. Now make a partial tiling by placing a domino on each cell in the top row and its neighbor, and place horizontal dominoes on the second row starting from the outermost cells. If we remove all cells that are covered by dominoes, the resulting region R' is balanced, and a jig-saw region with one less row. Eventually, we must arrive at the empty figure, and so by the partition theorem R must be tileable. □

Theorem 37 (Bougé and Cosnard (1992), Beauquier et al. (1995))**.** *A balanced stack polyomino is tileable.*[23]

[Not referenced]

Proof 1. We can delete a cylinder from the stack polyomino (Theorem 34) and the result a new stack polyomino (by Theorem 35) or the empty region. We can continue this process until the last region is empty. By Theorem 29 this means the region is tileable. □

[23] A fourth proof can be derrived from a more general theorem in Beauquier et al. (1995). Their result applies to tiling a stack polyomino with two bars, one horizontal and the other vertical, with no rotations allowed. If both bars have length two, we have the case of dominoes where any orientation is allowed. They introduce a lot of machinery to prove the general theorem. However, if it is made specific for dominoes, the ideas used resemble those of the second proof given here.

Proof 2. (Bougé and Cosnard (1992), Korn (2004, Part of Theorem 11.1, p. 156)) There must be at least two sides with length greater than two (Theorem 33), and so at least one side that is not the base. If this side is vertical, remove the two cells connected to this side and the top corner; if this side is horizontal, remove two cells connected to this side and the left or right corner. In all cases, the remaining figure is also a stack polyomino. Eventually, we must arrive at a single domino. Re-inserting dominoes in the reversed order in the same positions now yields a tiling of the stack polyomino. □

Proof 3. (Bougé and Cosnard, 1992) Suppose the top row has an odd number of cells. Then put horizontal dominoes on the top row to leave any cell not tiled. If we remove all cells tiled, we get a new region R' which is a balanced jig-saw region, which is tileable (Theorem 36). Therefor, R must be tileable. □

Theorem 38 (Thiant (2003))**.**

[Not referenced]

Coloring

WE MAY WONDER if other colorings exist that could give us information when the checkerboard coloring does not.

In a sense, the checkerboard coloring gives us the best information we can hope for in the general case. However, other colorings give us information in specific cases.

We first see what happens if we add a color.

Let us color a region with three colors, amber, blue, and cherry, such that colors in each row cycle, and diagonals have the same color. This type of coloring is called a **flag coloring**.

When we place a domino, it always covers two different colors, and there are three such arrangements. Let's denote the number of times a domino of each type is used by k_{AB}, k_{AC}, and k_{BC}, and the number of cells of each color in a region by A, B, and C. We have the following equations relating these values:

$$A = k_{AB} + k_{AC} \tag{6}$$

$$B = k_{AB} + k_{BC} \tag{7}$$

$$C = k_{AC} + k_{BC} \tag{8}$$

With a bit of algebra, we get the following solutions to the equa-

tions above:

$$k_{AB} = \frac{A+B-C}{2} \tag{9}$$

$$k_{AC} = \frac{A-B+C}{2} \tag{10}$$

$$k_{BC} = \frac{-A+B+C}{2} \tag{11}$$

These must all be whole numbers and non-negative (because, remember, they correspond to the numbers of dominoes), and this becomes a criterion: for a tiling to exist, it is necessary that k_{rb}, k_{ry} and k_{rb} are all non-negative integers.

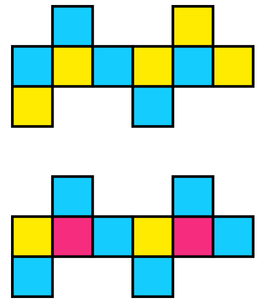


Figure 30: A region that is balanced, but does not satisfy another color criterion.

Figure 30 shows an example of a region that is balanced, but does not satisfy this new criterion.

If you experiment a bit with this criterion, you will find that the regions are quite pathological and in general it is usually easy to find out they cannot be tiled *without* using the criterion. What is going on?

From the equations, we can make the following observations: k_{xy} will be an integer unless the number of cells is odd. So this part is of little help. For k_{xy} to be negative, one of the colors must exceed the sum of the other two. And regions that does this tend to be spiky and uninteresting.

What if we changed the pattern? We have not used any detail of the pattern, except that neighboring cells cannot have the same color. If we color with only this restriction, it's rather easy to come up with a coloring where one color dominates. Figure 31 shosw this coloring applied to Figure 14(h).

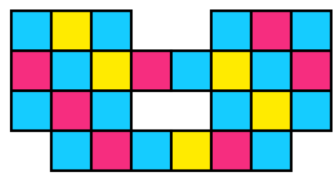


Figure 31: A coloring showing the region is not tileable.

But do we really need the two "looser" colors to be different? The answer is we don't. Let's set this up as we did before.

Color a region with amber and blue such that blue cells do not have any blue neighbors. We call such a coloring a **discriminating coloring**, with blue cells the **isolated** color. The tiles can now be of two types, blue-and-amber and amber only, and let's use k_{AB} and k_{AA} to denote how many of each tile we have.

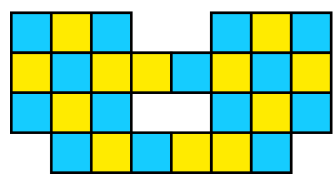


Figure 32: A coloring showing the region is not tileable. There are 14 blue cells, and 12 yellow cells. Since blue cells have no neighbors, if the region was tileable we would have at least at most as many blue cells as yellow cells.

Figure 32 shows this coloring applied to Figure 14(h).

We now have the following equations:

$$A = k_{AB} + 2k_{AA} \tag{12}$$

$$B = k_{AB} \tag{13}$$

Solving, we get

$$k_{AB} = B \tag{14}$$

$$k_{AA} = \frac{A-B}{2} \tag{15}$$

$$\tag{16}$$

For a tiling, k_{AB} and k_{AA} must be non-negative integers. k_{AB} will always be a non-negative integer; however, k_{AA} will be negative when $B > A$. The number k_{AA} will always be an integer as long as the number of cells is even.

If a region has a coloring for which k_{AA} is negative, we call the region **unfair** by that coloring.

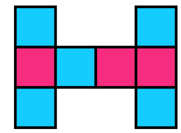


Figure 33: Double-T with alternative coloring

Example 10. *Color the double-T as shown in Figure 10. With this coloring, $B > W$, and since no black cells have black neighbors, we conclude the tiling is impossible.*

Problem 9. *For each remaining region in Figure 14, find a discriminating coloring by which it is unfair.*

We state this as a theorem that we will call the **general color criterion**. Note that it actually generalizes (and contains) the checkerboard criterion. However, the concept of balanced polyominoes is still useful, and when it works it's a very efficient way to describe the particular coloring.

Theorem 39 (Second color criterion)**.** *If there is a discriminating coloring by which a region is unfair, then the region not tileable by dominoes.*

[Referenced on page 31]

Proof. WLG let black be the isolated color. If a tiling exists, there must be b dominoes that cover a black and white cell, and w dominoes that cover only white cells. The number of black cells in the region is given by $B = b$, and then number of white cells $W = 2w + b$. Now if $B > W$, then $b > 2w + b$, and hence $2w < 0$, which is impossible (the number of white only dominoes must be positive or zero). Therefore, no such tiling exists. □

Problem 10. *Show that we cannot get more information by adding more colors using this type of criterion.*

In a sense, the new color theorem is really just the marriage theorem in disguise. The following theorem shows the connection between them.

Theorem 40. *A region has an unfair discriminating coloring if and only if it has a bad patch.*

[Referenced on page 31]

Proof. If. Color the region with the checkerboard coloring. If the region has a bad patch, it has a black bad patch. Let S be the biggest set that contains the black bad patch and is also a bad black patch.

Then $|\mathcal{W}(S)| < |\mathcal{B}(S)|$, and so $|\mathcal{W}(R-S)| \geq |\mathcal{B}(R-S)|$ (if it was not, we could make S bigger). Now swap all the colors in $R-S$. We now have a coloring in which there are more white cells than black cells, and all white cells only have black neighbors. Therefore we have an unfair discriminating coloring.

Only if. Suppose we have an unfair discriminating coloring of a region. Then the region is untilable (Theorem 39), and hence, in a checkerboard tiling, there is a bad patch (Theorem 26). □

From this theorem, we get the following:

Theorem 41. *A region is tileable if and only if all its discriminating colorings are fair.*

[Not referenced]

Proof. If. Suppose all the discriminating colorings of a region are fair, but it is untilable. Then, by Theorem 26 there is a bad patch, and by Theorem 40 there exists an unfair discriminating coloring, which contradicts our initial assumption, therefore, the region must be tileable. *Only if.* Suppose a region is tileable. If it had a unfair discriminating coloring, it would not be tileable (by Theorem 39), and so all discriminating colorings must be fair. □

At the beginning of this section, we said that the checkerboard coloring gives us the best information in the general case. The reason for this is that of all discriminating colorings, the checkerboard has the highest density of isolated color, and can therefor discriminate the most figures (of a certain area). I am not going to make this idea more precise here. Furthermore, it makes ideas such as flow, and (as we will see in another essay) height functions possible, which other discriminating colors do not.

The Internal Structure of Tilings

SOME TILEABLE REGIONS HAVE cells that are tiled the same way in any tiling of the region. We call such cells **frozen**.

As we will see, there are three ways in which cells can become frozen:

- When it is part of a region that can geometrically only be tiled one way (when it is part of a peak, a notion we shall define shortly).
- When other placements violate the border crossings theorem 14.

- When other placements violate the flow theorem (Theorem 18).

We will see that cells that are not frozen are always part of some cycle, and that by rotating the cycle we can find a new tiling of the region. We will look at how we can put cycles together to build regions with a specific number of tilings, which will lay the groundwork for counting tilings in the next section.

A key point from this section is the following: any two tilings are connected through a series of simple operations that involve rotations on cycles of dominoes.

Cycles

A **path** is a sequence of lattice points $v_1, v_2, \dots, v_n$ such that v_{i+1} and v_i are neighbors, and no cell is repeated in the sequence.

A **strip polyomino** is a polyomino whose cells form a sequence $v_1, v_2, v_3, \dots v_n$ such that v_i always neighbors v_{i+1} and no cell is repeated in the sequence. If the we also have v_n neighbors v_1, then we call the strip polyomino **closed**.[24]

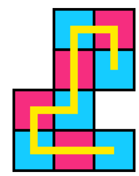


Figure 34: Strip polyomino

[24] In Beauquier et al. (1995) the word *ring* is used for a closed strip.

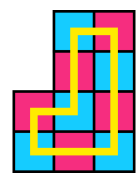


Figure 35: A closed strip

Theorem 42 (Beauquier et al. (1995)). *Strip polyominoes with even area are tileable. Closed strip polyominoes with even area has eat least two different tilings.*

[Referenced on pages 33, 34 and 36]

Proof. The cells of the strip polyomino forms a path $v_1, v_2, \dots, v_n$, with n even. We can therefore partition the cells into sets $\{v_1, v_2\}$, $\{v_3, v_4\}$, ..., $\{v_{n-1}, v_n\}$, each containing two elements. Since in each set the two cells are neighbors, the set can be tiled by a domino, and so the entire region can be tiled by dominoes.

If the strip is open, then another tiling is given by $\{v_n, v_1\}$, $\{v_2, v_3\}$, ..., $\{v_{n-2}, v_{n-1}\}$. □

Theorem 43 (Beauquier et al. (1995)). *If we apply a checkerboard coloring to a strip polyomino, and its two ends have opposite colors, the path has an even number of cells, otherwise it has an odd number of cells.*

[Referenced on page 38]

Proof. We note that it is true for a strip polyomino with area 1. Now suppose it holds for strip polyominoes with area k. Consider now a strip polyomino with area $k+1$, and suppose its two ends have the same color. Now remove one end, to get a path of length k. Since the new end must have opposite color of the one removed, the two ends are now of opposite color, and hence n must be even, and so $k+1$

must be odd. Suppose then the two ends have different colors. If we remove one end from the strip polyomino, the new polyomino must have ends of the same color, and hence k must be odd, and so $k+1$ must be even. We proved that the theorem also holds for $k+1$, and therefore, it must hold for all k. □

A **peak** is a cell of a region with only one neighbor in the region.

A **snake** is a polyomino that has two peaks, and every other cell has exactly two neighbors Goupil et al. (2013). Snakes are strip polyominoes, and therefore snakes with even area are tileable.

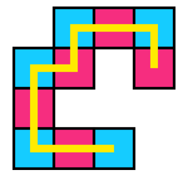


Figure 36: Snake

Theorem 44. *All even-area rectangles are tileable.*

[Not referenced]

Proof. If the area of a rectangle is even, then either its width or height must be even. WLG say the width is even. Now partition the rectangle into rows. Each row is a snake with even area, and therefore tileable (Theorem 42), and so is the whole region (Theorem 2). □

k	$P(k)$ A000105	$S(k)$ A002013
1	1	1
2	1	1
3	2	1
4	5	2
5	12	3
6	35	7
7	108	13
8	369	31
9	1285	65
10	4655	154

Table 2: Number of free snakes $S(k)$ compared to the number if free polyominoes $P(k)$.

In fact, rectangles are strip polyominoes, and what is more, rectangles with even area are closed strips.

Theorem 45. *Even-area rectangles as closed strips.*

[Not referenced]

Proof. Let the width and height of the rectangle be m and n. WLG suppose the width is even. Partition the rectangle into $m+1$ snakes as follows: form the first m snakes S_i from all cells in each column except the top one; form the last snake S_{m+1} from the cells in the top rows. To each strip assign a head and tail: Except for the last snake, odd numbered snakes get a tail at the top and head at the bottom. Even-numbered snakes get a head at the bottom and a tail at the top. The last snake gets a head at the left and a tail at the right.

Now if we join the snakes head to tail in order, and join the head of the last to the tail of the first, we have a path through all the cells in the rectangle, which proves it is a closed strip. □

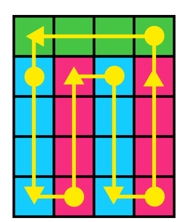


Figure 37: A rectangle is a strip polyomino.

A **ring** is a polyomino where each cell has exactly two neighbors. Rings are closed strip polyominoes.

Rings are closed strips, and therefore they have even area, and exactly two tilings.

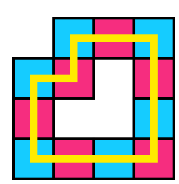


Figure 38: Ring

Although we prove theorems for strip polynomials, it is not in general easy to determine whether a given polyomino is a strip polyomino (for example, it is not immediately obvious that Figure 14(e) is not a strip polyomino). On the other hand, it is easy to recognize snakes and rings.

Problem 11. *Define the dilation of a region as the region with all cells of the original region and their neighbors. Let R' be the dilation of R. When is $R' - R$ a ring?*

Problem 12. *Define the erosion of a region as the region with all cells of the original region that has four neighbors. Let R' be the erosion of R. When is $R' - R$ a ring?*

Problem 13. *What if in the definitions in the above we use 8-neighbors instead of 4-neighbors?*

Theorem 46 (Gomery's Theorem)**.** *If we remove a white and black cell from a checkerboard colored strip polyomino, the remaining region is tileable by dominoes.*[25]

[25] The theorem, by Ralph Gomery, was originally given in only for the 8×8 square(Honsberger, 1973, p. 65–67), and the proof was slightly different. However, the main idea of the proof is the same as given here.

[Not referenced]

Proof. Let the cells along the path of the strip polyomino be v_1, v_2, v_3, $\ldots$, v_k, and suppose the removed cells are v_m and v_n with $m < n$.

If the removed cells are consecutive in the path (Figure 39), then the region has a path $v_{n+1}, v_{n+2}, \ldots v_k, v_1, v_2, \ldots, v_{n-2}$. This is an even strip, and so is tileable (Theorem 42).

If the removed cells are not consecutive in the path (Figure 40), they partition the strip into two strips. The ends of each strip neighbor cells of opposite color, and therefore the ends of each strip is of opposite color. Therefore, the strips are tileable, and so is the whole region.

□

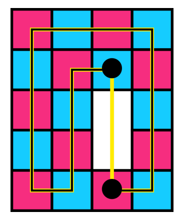


Figure 39: Removing two cells consecutive in the path.

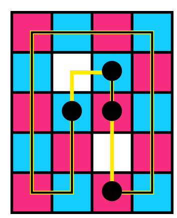


Figure 40: Removing two cells not consecutive in the path.

Frozen Cells

In Figure 42, we can see a domino can fit only one way in the two yellow cells. We can then remove these cells, and consider whether the rest of the region can be tiled. Since this is a 2×2 square, it can be, and so we conclude the whole region is tileable. The same technique works for more complicated regions. In Figure 41 we show the process for a more complicated region.

When you apply this technique to Figure 14(b), you get a disconnected monomino after the first step, which is not tileable, and so the whole region is not tileable.

We will now formalize this process in a series of definitions and theorems.

A subregion of R is **frozen** if it can be covered by dominoes in only one way in any tiling of R.

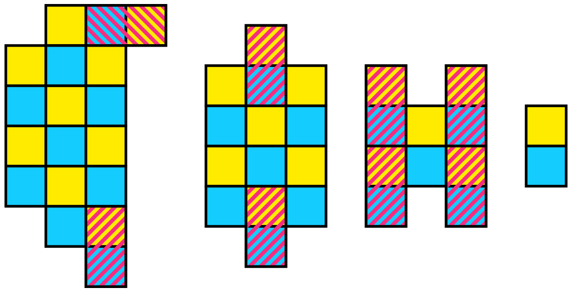


Figure 41: Illustrating peak removal.

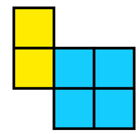


Figure 42: The yellow cells can only be covered in one way by a domino.

Theorem 47. *Suppose a region R is partitioned into two partitions, S_1 and S_2, and S_1 is tileable, and would be frozen in any tiling of R if it exists. Then the region is tileable if and only if S_2 is tileable.*

[Referenced on page 36]

Proof. If. If S_2 is tileable, then R is tileable by Theorem 2.

Only if. Suppose R is tileable. Because dominoes can cover the frozen partition in only one way, in a tiling of R the dominoes that are part of S_1 must also tile S_1, since it is indeed tileable. Therefore, the dominoes *not* part of S_1, must tile S_2. □

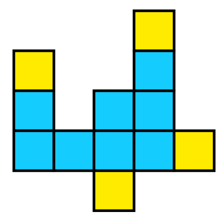


Figure 43: A region with 4 peaks, shown in yellow.

Theorem 48. *In a tileable region, a peak and its neighbor are frozen.*

[Not referenced]

Proof. There is only one way for the domino to cover the peak, and in that one way it also covers the neighbor. So if the region has a tiling, the neighbor cell can also only be covered one way. □

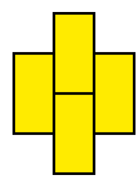


Figure 44: A region whose compact subregion is empty.

From this theorem, it follows that a region with a peak is tileable if and only if the region(s) that remain after we remove a peak and its connected cell. We can repeatedly remove peaks and their connected cells until no peaks remain. The resulting region (which may not necessarily be connected) is called the **compact subregion**. The compact subregion of R is denoted R^*. Note that R^* can be empty (Figure 44), or disconnected (Figure 45). It is possible that $R^* = R$, in which case we call R **compact**.

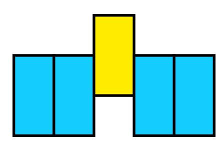


Figure 45: A region whose compact subregion (blue) is disconnected.

Problem 14. *Suppose R has no holes. Prove R^* does not have any holes.*

Problem 15. *What is the bound on $\Delta(R)$ as a function of the number of cells in R of R is compact?*

Theorem 49. *A region R is tileable if and only if its* ***compact subregion*** *R^* is tileable.*

[Not referenced]

Proof. Let $R' = R - R^*$, so that R' and R^* are partitions of R. Since all the cells in R' are frozen, by Theorem 47 R is tileable if and only if R^* is tileable. □

If a region consists out of disconnected parts, clearly, it is tileable if and only if each part is tileable (this is also a consequence of Theorem 2.) The tileability of all regions thus boils down to whether connected compact shapes are tileable.

Theorem 50 (Beauquier et al. (1995))**.** *If a region has a unique tiling, it must have at least two peaks.*

[Referenced on page 39]

Proof. Suppose a region has no peaks, and that it has a tiling. Now construct a sequence of cells as follows: v_1 is any cell, and v_2 is the cell in the same domino. Choose v_3, a neighbor of v_2 that is not v_1 (we can do this, since v_2 is not a peak).

We can continue in this fashion, always choosing new cells that haven't been chosen before, until eventually, we are "trapped" or run out of cells. Now the last cell, v_n, must have a neighbor that is not v_{n-1}, and is already part of the sequence, say v_i. (If it did not, then v_n would be a peak). So we have a sub-sequence $v_i, v_{i+1}, \dots v_n$ that is a closed strip. Closed strips are tileable in more than one way (Theorem 42), and therefore, the entire region must be tileable in more than one way.

Suppose the region does have one peak. Construct the same sequence as above, but choose v_1 as the peak. □

This implies that compact shapes have at least two tilings.

If we recursively remove peaks and their connected cells, we are eventually left with either a compact region, or the empty region. If the former is the case, the region is uniquely tileable. Otherwise, it has more than one tiling.

Tiling Transformations

In this section we look at how one tiling can be transformed into another with a series of "primitive" transformations.

Theorem 51. *A cell is not frozen if and only if it is part of a tiled closed strip.*

[Referenced on page 38]

Proof. If. Suppose a cell v_1 is part of a tiled closed strip, and its neighbor in the the same domino is v_0, and its other neighbor along the strip is v_1. Then by Theorem 42 there is another tiling of the strip, and in that tiling v_1 and v_2 are in the same domino. Therefore, there is more than one way for v_1 to be covered by dominoes, and therefore it is not frozen.

Only if. Suppose a cell v_1 can be tiled in two ways. In one tiling v_0 is the same domino, and in the other tiling v_2 is in the same domino. Then let v_3 be v_2's neighbor in the first tiling, v_4 is v_3's neighbor in the second tiling, and so on. We can always extend the sequence, but since there are only a finite number of cells in the region, eventually the next cell must be one that is already in the sequence. Moreover, this repeated must be v_0, since a cell can be in the same domino in two different tilings at most two times, and any other cells v_i is already in the same domino as either v_{i-1} and v_{i+1}.

Therefore we have a sequence of cells $v_0, v_1, \ldots, v_k$, where v_{i+1} is a neighbor of v_i, and v_n is a neighbor of v_0. Therefore, the cells $v_0, v_1, \ldots, v_n$ form a closed strip. □

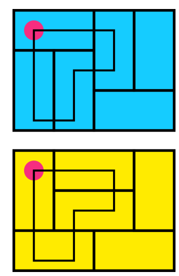


Figure 46: An example of a closed strip that contains a point and is tiled in both tilings.

The regions in Figure 47 all have frozen cells that cannot be part of any tiled closed strip.

Problem 16. *Give an example of a region with no frozen cells that has two cells that are not in the same tiled closed strip.*

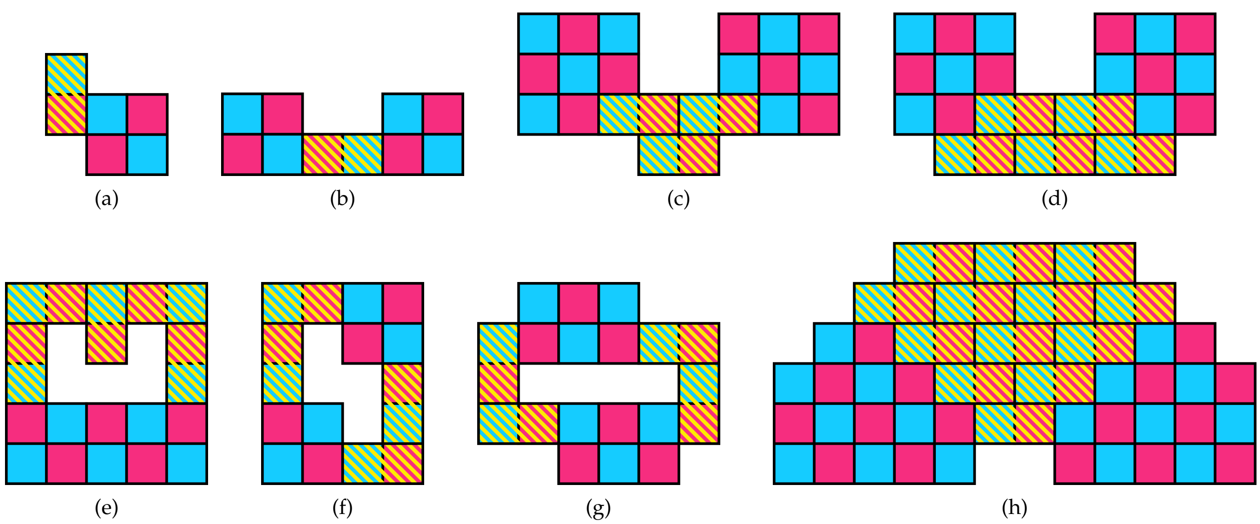


Figure 47: Examples of regions with frozen cells, marked with yellow stripes.

Theorem 52. *If two closed strips are put side-to-side, then together they form a strip polyomino.*

[Not referenced]

Proof. Let the two closed strips be $u_1, u_2, \ldots, u_m$, and $b_1, b_2, \ldots, b_n$, and suppose u_i is neighbors with b_j. Then a strip is given by

$$u_{i+1}, u_{i+2}, \ldots u_m, u_0, u_1, \ldots, u_i, v_j, v_{j+1}, \ldots, v_n, v_0, v_1, \ldots v_{j-1}.$$

□

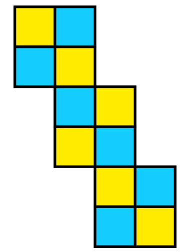


Figure 48: An example of 3 connected closed strips that do not form a strip.

Figure 48 shows an example of 3 connected closed strips (2×2 squares) that are connected, but do not form a strip.

An analogous argument shows any non-frozen cell tiled by a vertical domino, if it has a non-frozen horizontal neighbor, it can also be tiled horizontally.

In a region, if we replace a closed strip with its dual tiling, we call this operation a **strip rotation**[26] (Figure 49). When the closed strip is a 2×2 square, we call the strip rotation a **flip**. In Theorem 51 we saw how to construct a closed strip that contains a cell from two different tilings. In the theorem below, we exploit this idea to prove that one tiling can be transform into any other tiling through a sequence of strip rotations.

[26] In Propp (2002)the other calls this operation (in the context of oriented graphs), a *face twist*.

Theorem 53 (The tiling connection theorem)**.** *We can obtain one tiling from another by a sequence of strip rotations.*

[Referenced on page 43]

Proof. Pick any cell with a different tiling in the two tilings. Construct the closed strip as in Theorem 51, and perform a strip rotation on that strip in the first tiling. All the dominoes in that strip now has the same tiling as in the second tiling. Repeat the process. Notice, that subsequent rounds do not change any dominoes that are already in the same position as in the second tiling. Since the region is finite, and we reduce the number of dominoes that differ from the second tiling in each step, the process must end when all the dominoes match. □

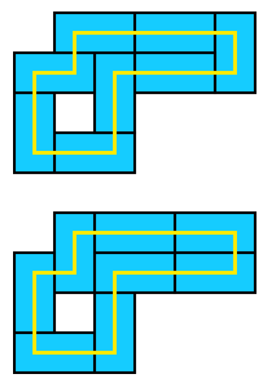


Figure 49: An example of a strip rotation.

Theorem 54. *Let R be a tiled region, and S a subregion of R. Performing a strip rotation on R does not affect the flow on S.*[27]

[Not referenced]

[27] These ideas are more or less presented in Saldanha et al. (1995). They consider flow through cuts that do not disconnect the region.

Proof. This is clearly true when the closed strip is entirely within S or entirely out of S.

Suppose then the closed strip goes over the border of S.

If a strip outside S meets S at cells u and v, then either the strip has an even number of cells (and so u and v must have opposite colors, by Theorem 43), or an odd number of cells (and u and v must have the same colors, by Theorem 43).

In the first case, dominoes must either cross S at both points, or neither, and the net contribution of dominoes at those two points are zero. Doing a strip rotation will cause the opposite; either dominoes don't cross at either point, or they do at both. Again, the net contribution to the flow is 0.

In the second case, a domino must cross at the one point, and not the other, and therefore the contribution to the flow is $+1$ or -1, depending on the color of the cell where the domino crosses inside S. Say the crossing happens at u, and that u is black. Then the domino contributes +1 to the flow. v must be white too. When we do a rotation, there is no crossing at u, and a crossing at v. Since v is white, the contribution to the flow is 1. A similar argument shows when u is white the contribution before and after the strip rotation is -1.

So in all cases, for every strip, part of the closed strip, that meats S, we have that their contributions to the flow stays unchanged by strip rotations, and so the overall flow is unchanged. □

Theorem 55. *Every tileable region contains at least one of the following:*

- *Two peaks.*
- *A* 2×2 *square subregion.*
- *A hole.*

[Referenced on page 39]

Proof. Suppose the region has a unique tiling. Then it contains two peaks by Theorem 50. Suppose then the region has more than one tiling. Pick a cell v that is not frozen. This cell must be part of a closed strip. Inside this closed strip, choose the left-most bottom-most cell. Because it is part of a closed strip, it must have at two neighbors inside the strip, and since this is a left-most bottom-most cell, it must have a top neighbor v_T and right neighbor v_R. Now consider the position u to the top of v_R. Either there is a cell, or there is not. In the former case, we have a 2×2 square. Suppose then there is not a cell in that spot. Since all the cells in the closed strip cannot lie to the left of v, they must surround u, and therefore there is a hole. □

The following theorem allows us to do induction on closed strips: it gives us a way of breaking closed strips into smaller closed strips.

Theorem 56. *Every closed strip without holes and more than 4 cells contains a smaller tiled closed strip as subregion.*

[Not referenced]

Proof. By Theorem 55 a closed script must have a either a hole or a 2×2 square as subregion, and since this closed script does not have a hole, it must contain a 2×2 subregion.

Now consider how that square is tiled. Since the flux is 0, we must have that 0, 2, or 4 dominoes overlap the border.

- If 0 dominoes overlap the border, we have a flippable pair, which is a tiled closed strip, and a subregion smaller than the original.
- If 2 dominoes overlap, the cells where dominoes overlap must be neighbors. There are four ways in which we can form a closed strip with this configuration (Figure 50). In two of them the branches of the strip must overlap (Figure 50(a) and (b)), and in one of the remaining the two branches are untileable (Figure 50(c)). That leaves only one configuration (Figure 50(d)).

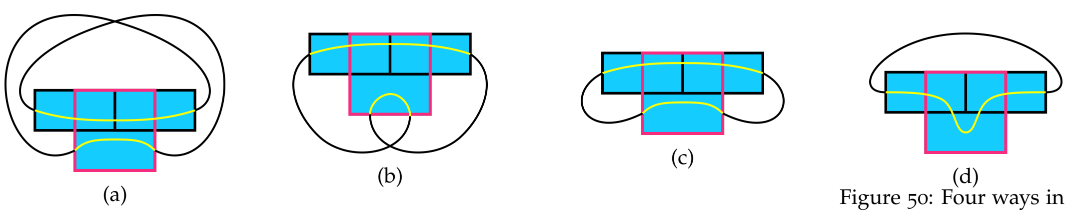


Figure 50: Four ways in which closed strips can be formed when 2 dominoes overlap.

  In this configuration we can form a smaller closed strip by removing the domino that lies completely within the square (Figure 51).

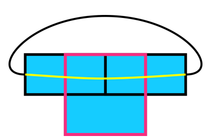


Figure 51: How a smaller closed strip can be formed this configuration.

- If 4 dominoes overlap, there are several possibilities of forming a closed strip, some are shown in Figure 52. Most of these have branches that cannot be tiled, (Figure 52(a)), or leave out one of the dominoes (Figure 52(b)); many also have branches that intersect. Two cases work (Figure 52(c) and (d)).

  In these cases we can split the path into two as shown in Figure 53. Either path is shorter than the original.

□

Theorem 57. *A tiling of a closed strip without holes must have at least one flippable pair.*

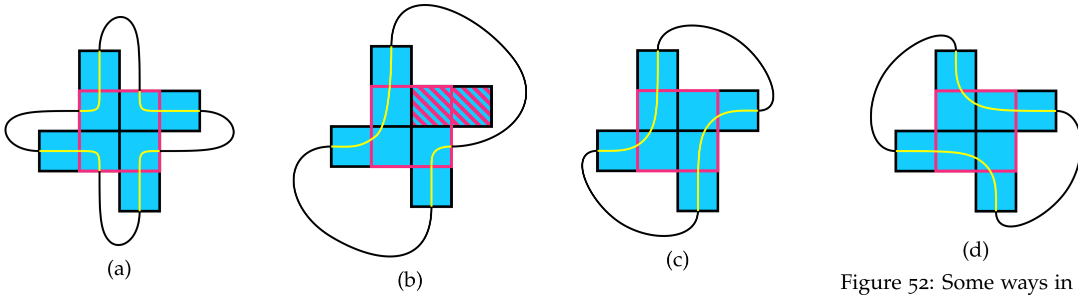


Figure 52: Some ways in which closed strips can be formed when 4 dominoes overlap.

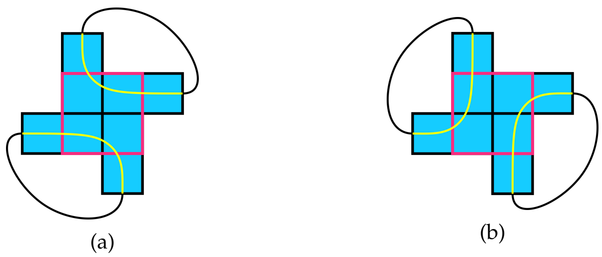


Figure 53: Splitting paths.

[Not referenced]

Proof. If the closed strip is a 2×2 square, the closed strip is a flippable pair and we are done.

Otherwise, by Theorem theorem:closed-strip-decomposition, the region contains a smaller closed strip as subregion. Continue to find a smaller and smaller subregion. Since the tiling is finite, eventually we must find a 2×2 square, which is a flippable pair. □

Theorem 58. *A strip rotation (of a strip without holes) is equivalent to a sequence of flips.*

[Not referenced]

Proof. It is true for a closed strip of 4 cells.

Find a flippable pair. If it is a 2×2 square, a flip rotates the entire strip.

Otherwise, the closed strip has more than 4 cells, and it can have either one branch (Figure 54(a)) or two branches (Figure 55(a)) to complete the strip.

If it has one branch. Do the flip (Figure 54(a)). One domino is now in the right position, the other, with the branch, makes a closed strip with less than n cells, so we can do a rotation by induction. After this rotation, the complete strip is rotated.

Figure 54: A strip with one branch before and after the flip.

If it has two branches. Do the flip. We can now form two closed strips with one domino in each (Figure 55(a)). Each of these can

be rotated by induction. After the rotation of each, the full strip is rotated too.

Figure 55: A strip with two branches before and after the flip.

□

Theorem 59. *If a closed strip has a filled interior, a rotation of the strip is equivalent to a sequence of flips.*

[Referenced on page 43]

Proof. Find a domino with it's long edge on the border of the interior (such a domino exist by Theorem 33). The long edge on the border neighbors the outer closed strip in two cells; either they lie in the same domino, or they don't.

If they don't, they lie in two dominoes, who may or may not be neighbors in the strip. If they are, we can form a new closed strip by simply including the interior domino between them. Otherwise, we form two strips, a closed one with the interior domino between them, and the other what used to be between the two outer dominoes. These are closed if there is more than one domino.

If they do, we flip the pair, and make two new closed strips with one in each.

We continue this process (in each case using either of the available closed strips constructed so far) until the entire interior is enclosed within one of the closed strips $R_1, R_2, \ldots$. We may have shedded several strips $S_1, S_2, \ldots$ in the process.

Now each of R_i is a closed strip without holes, so a rotation on these are equivalent to flips. Perform the rotation on each.

For each of S_i, they may be a closed strip without holes, or a single domino. In the case of the former, a rotation is equivalent to a sequence of flips, so we can do a rotation. In the case of the latter, after all the rotations, one of the dominoes that used to be a neighbor inside the strip is now a flippable pair with it. Do the flip.

All the dominoes in in the entire closed-strip are now rotated. Some dominoes in the interior might be in wrong positions, so they need to be restored. Pick a cell, and find a closed strip from the current and desired tiling. If this strip has an empty interior, we can perform a rotation equivalent to flips. Otherwise, we re-apply this theorem (it must end, because with each re-application we have fewer tiles and there is only a finite amount.) We repeat until the interior is completely restored.

We have performed a rotation of the out strip, with only using flips and flip-equivalent rotations. Therefore, the entire operation is flip-equivalent. □

Theorem 60 (Saldanha et al. (1995)). *We can obtain one tiling of a region without holes from another by a sequence of flips.*

[Not referenced]

Proof. Any rotation of a strip with a filled interior is equivalent to flips (Theorem 59), and any tiling can be transformed into another by a series of rotations (Theorem 53). Therefore, any tiling can be transformed into another with a series of flips.[28] □

[28] See also Rémila (2004) which gives an easier treatment of flips and domino tilings.

Further Reading

In Mendelsohn (2004) the author gives a gentle introduction to the relationship between domino tilings and graph theory.

In other essays we will use a variety of color arguments. See Engel (1998) for other applications and ways to structure coloring arguments.

As mentioned in a side note, the marriage theorem we presented here is a specific example of the much more general theorem in the area of *matching theory*. For a survey on the development of matching history, see Plummer (1992), and for a detailed treatment, see the book Lovász and Plummer (2009). Many books on combinatorics and graph theory contain chapters on matching, see for example Harris et al. (2008), Diestel (2000) and Bondy and Murty (1976).

Domino tilings and their statistics are of interest to physicists because they model the behavior of certain types of molecules. In this context, dominoes are usually called *dimers*, the tilings may be considered on more general graphs than grids. The statistical model that deal with these tilings (or *coverings*) is called the *dimer model*. For a survey on the dimer model, see Kenyon (2000). In the paper Cohn et al. (2001) the authors give a detailed analysis of the statistics of arbitrary regions.

For a less naive view on the topics discuss here, I recommend the following sequence of papers to get familiar with the algebra of polyomino tilings:

(1) Propp (1997) is a gentle introduction to Conway's work,

(2) Conway and Lagarias (1990) is the paper where Conway introduces some group theoretic tools to study tilings, and

(3) Thurston (1990) is where height functions are introduced for the first time, and a polynomial time algorithm is described.

The thesis Donaldson (1996) discusses these ideas in a more leisurely fashion, and among other things, works through a proof of Thurston's algorithm.

For domino tilings, specifically, the following papers are a useful start[29]:

[29] I will give a more expansive bibliography after we covered more topics. The papers listed here deal more or less with the same topics than this essay

- *Tiling figures of the plane with two bars* (Beauquier et al., 1995)
- *Spaces of domino tilings* (Saldanha et al., 1995)
- *The lattice structure of the set of domino tilings of a polygon* (Rémila, 2004)
- *Optimal partial tiling of Manhattan polyominoes* (Bodini and Lumbroso, 2009)

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