Pentagons

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Contents

1 Notation

1.1 Standard labeling

I use the standard notation $ABCDE$ for a pentagon with the vertices A, B, C, D and E. The five lines AD, BE, etc. are called the **diagonals** of the pentagon. Diagonals intersect in P, Q, R, S and T, with P opposite A, Q opposite B, etc.

The diagonals divide each vertex angle into three **subangles**. The three subangles of vertex A is denoted A_1, A_2 and A_3, and similarly for the other vertices. The five triangles ABC, CDE, etc. are called **vertex triangles**; the five triangles ABD, BCE, etc. are called **edge triangles**.

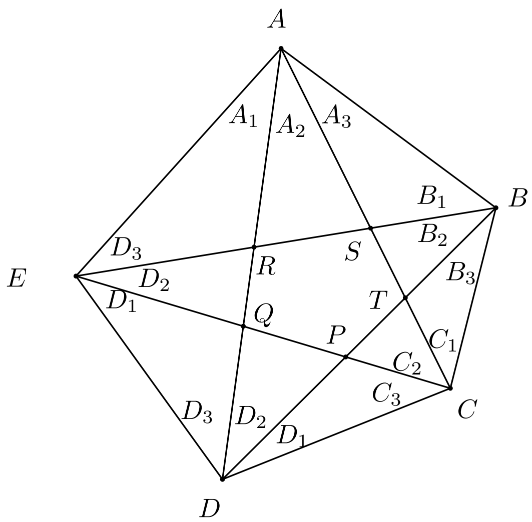

Figure 1: Standard labelling of a pentagon.

1.2 Cycle notation

Cycle notation can be used as shorthand sets of expressions or equations that apply (symmetrically) to all vertices in a set.

For example, the relation

$$A = \frac{B+C}{2} \quad \circlearrowleft ABCDE$$

is shorthand for the following five equations:

$$A = \frac{B+C}{2}$$
$$B = \frac{C+D}{2}$$
$$C = \frac{D+E}{2}$$
$$D = \frac{E+A}{2}$$
$$E = \frac{A+B}{2}$$

Because this document deals with pentagons, most of which are denoted $ABCDE$, I will drop the five vertices from the notation if it is clear what is meant. The above then simply becomes

$$A = \frac{B+C}{2} \quad \circlearrowleft .$$

In essence, it is a more systematic way of writing "etcetera".

With this notation, we can express the fact that a pentagon is equilateral with

$$AB = BC \quad \circlearrowleft,$$

or equiangular with

$$A = B \quad \circlearrowleft .$$

The notation can also be used to cycle over two sets of vertices. For example, the expression

$$AS = ST = TC \quad \circlearrowleft ABCDE, PQRST$$

is shorthand for the following:

$$AS = ST = TC$$
$$BT = TP = PD$$
$$CP = PQ = TE$$
$$DQ = QR = RA$$
$$ER = RS = SB$$

Since we will mostly deal with the second set of vertices being the diagonal intersections of the pentagon $ABCDE$, I omit the second set as well when it is clear what is meant. The above then simply becomes

$$AS = ST = TC \quad \circlearrowleft$$

I also use the cycle symbol in sums. For example,

$$\sum_{\circlearrowleft ABCDE} AB = AB + BC + CD + DE + EA$$

When the vertex being used is clear from the context, I will omit it. The sum above is then simply written:

$$\sum_{\circlearrowleft} AB$$

We define the product over a cycle similarly, for example:

$$\prod_{\circlearrowleft} AB = AB \cdot BC \cdot CD \cdot DE \cdot EA$$

Cycle statements are equivalent when we move all vertices to the next k-th vertex. For instance, these mean the same thing:

$$f(A,B,D) = g(A,C,D) \quad \circlearrowleft$$
$$f(C,D,A) = g(C,E,A) \quad \circlearrowleft$$

In sums and products a lot of manipulations is possible through re-arrangement. For instance

$$\begin{aligned}\sum_{\circlearrowleft} AB - AC &= AB - AC + BC - BD + CD - CE + DE - DA + EA - EB \\ &= AB - EB + BC - AC + CD - BD + DE - CE + EA - DA \\ &= \sum_{\circlearrowleft} AB - EB\end{aligned}$$

The final use of cycle notation is to denote sets. For instance, we may say "the lines AP $\circlearrowleft$ are concurrent", which simply means the lines AP, BQ, CR, DS, and ET are all concurrent. Here we left out the two vertex sets, as we usually do.

1.3 Area

The area of a polygon $XY \cdots Z$ is denoted $\mathcal{A}(XY \cdots Z)$.

2 Five points in a plane

Five points in a plane can be connected in interesting ways to form pentagons. Figure 2 shows a list of representative from 11 classes of pentagons. We only consider **proper pentagons**—those pentagons with no three vertices colinear, which also implies that all vertices are distinct.

Below is a rough, informal characterization of these classes. In all cases the vertices are classified by the angle on the coloured side. The characterization can be used to recognise pentagons visually, but a more technical characterization is necessary for proofs about classes of pentagons. I use the term "hole" very loosely to mean a patch where the polygon can be considered to overlap itself.

Class 1 Convex

Class 2 One concave vertex

Class 3 Two adjacent concave vertices

Class 4 Two non-adjacent concave vertices

Class 5 One intersection

Class 6 One intersection, one concave vertex (opposite intersection)

Class 7 One intersection, one concave vertex (adjacent to intersection)

Class 8 One intersection, two concave vertices (has a hole)

Class 9 Two intersections

Class 10 Three intersections, one concave vertex (has a hole)

Class 11 Five intersections (has a hole)

Classes 1-4 are simple pentagons, while the remainder or complex pentagons.
Theorem 1. *A pentagon (with appropriate non-degenerate conditions) can intersect itself zero, one, two, three or five times, but not four times.*

Proof. Figure 2 provides examples of pentagons that intersect themselves zero, one, two, three and five times.

The proof that pentagons cannot intersect themselves four times is quite technical, and requires some extra terminology. Two edges are **adjacent** when they share a vertex. Two distinct points are **joined** by an edge if they are the endpoints of that edge. Two points A and B are **connected** by a set of edges if

(1) the set has a single edge, and it joins the points, or

(2) the set of all edges but one connects A with a third point C, and the remaining edge joins the AC.

First note that no edge can intersect more than two other edges, for it cannot intersect with itself or the two adjacent edges, so it can intersect with only the other two edges.

Second, if a pentagon has four selfintersections, then not all edges can intersect with only one other edge. For clearly, the maximum number of total intersections possible when five edges have each at most one intersection is two.

Therefor, a pentagon with four intersections must have at least one edge with two intersections.

So let AB intersect two other edges. These two edges must be adjacent, for if they are not, then there are six edge endpoints that needs to be connected to form a pentagon. But at least three edges is required for this, and only two remain.

So let the two edges be CD and DE, sharing vertex D. Then either AC must be joined, or AE must be joined:

(1) If AC is joined, then BE must be joined, giving a pentagon with only two intersections in total.

(2) Instead, if AE is joined, then BC must be joined. Two cases are possible: either AE intersects CD, or it does not.

(a) If AE intersects CD, then either BC must intersect both AE and DE, or neither, giving either five or three total intersections.

(b) If AE does not intersect CD, then BC must intersect DE, and no other edge. This gives a pentagon with three intersections.

Therefor, no configuration is possible that gives four points of intersection. □

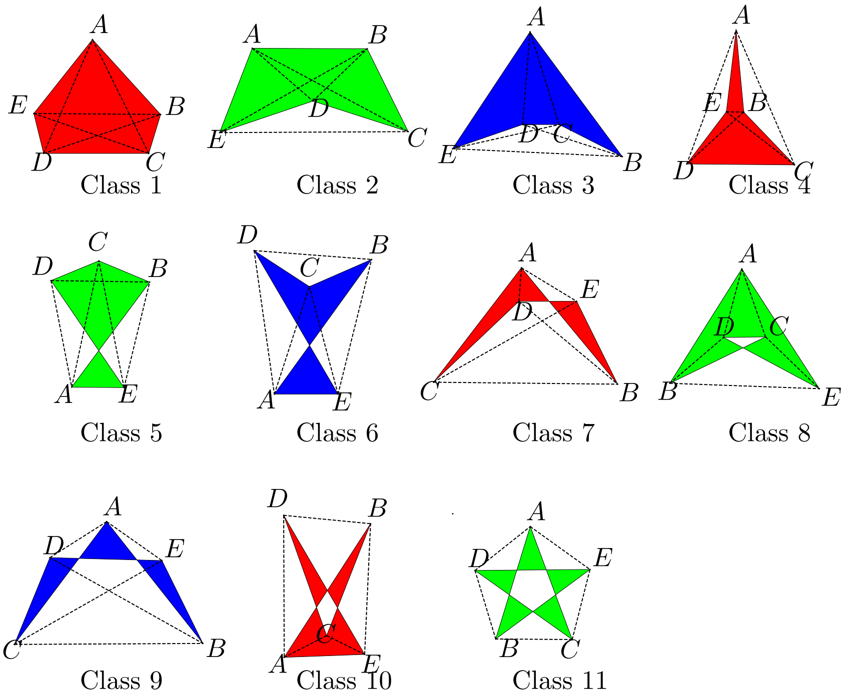

Figure 2: Pentgon classification

Definition 1 (Dual of a pentagon)**.** *If $ABCDE$ is a pentagon, then the pentagon $ACEBD$ is called the dual of that pentagon.*

The dual pentagon is precisely the pentagon whose edges are the diagonals of the original pentagon. The fact that pentagons have five diagonals is a nice coincidence which makes the idea of a dual figure natural. The idea needs modification to be applied to other polygons.

Theorem 2. *If pentagon $\mathcal{P}_1$ is the dual of $\mathcal{P}_2$, then $\mathcal{P}_1$ is the dual of $\mathcal{P}_2$.*

Proof. Let $\mathcal{P}_1 = ABCDE$. Then, since $\mathcal{P}_2$ is its dual, $\mathcal{P}_2 = ACEBD$. Then The dual of $\mathcal{P}_2$ must be $AEDCB$, which is the same pentagon as $ABCDE$, i.e. the dual of $\mathcal{P}_2$ is $\mathcal{P}_1$. □

Since the sides of a pentagon cannot intersect four times, it follows that there is no pentagon whose diagonal segments intersect four times (because if there were, its dual would have sides that intersect four times).

In general, the dual of a pentagon of a class can be in more than one class. Figure 3 shows that duals of Class 2 can be Class 2 or Class 5, and Table 1 summarizes the possible dual classes for each class. (Note, the table has been obtained experimentally, so it is possible there are some ommisions).

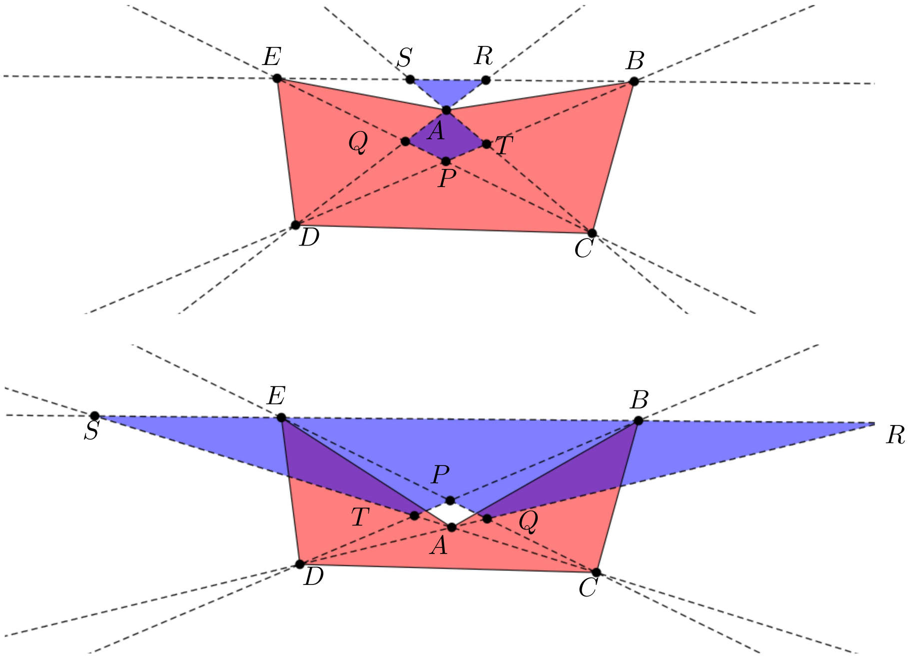

Figure 3: Duals

Class	Possible Classes of Duals
1	11
2	2, 4, 5, 7, 9, 10
3	5
4	2, 8
5	2, 3, 7, 9
6	2
7	2, 5
8	4
9	2, 5
10	2
11	1

Table 1: Classes for Duals

It should be clear that we should be able to split classes so that the dual of a pentagon in one class is always in just one class.

Definition 2 (Diagonal pentagon of a pentagon)**.** *The diagonal pentagon of pentagon $ABCDE$ is the pentagon $PQRST$.*

Class	Possible Classes of Diagonal Pentagons
1	1
2	2, 5
3	1
4	2, 7, 9
5	7, 8, 9
6	1, 3, 6
7	2
8	2
9	5, 7, 9, 10
10	4, 7, 9
11	2, 5, 11

Table 2: Classes for Diagonal Pentagons

Theorem 3. *If a pentagon is convex, it contains all the vertices of its diagonal pentagon.*

Proof. If $ABCDE$ is convex, then so is the quadrilateral $ABCE$. The diagonals of a convex quadrilateral intersect inside the quadrilateral at S. Hence, S is inside $ABCE$, and hence it is inside $ABCDE$ (since D is outside $ABCE$). Thus we have

$$S \text{ is inside } ABCDE \ \circlearrowleft,$$

and so all of $P \ \circlearrowleft$ is inside $ABCDE$. □

Theorem 4. *In a convex pentagon,*

$$AR < AQ \ \circlearrowleft .$$

Corollary 5. *If a pentagon is convex, so is its diagonal pentagon.*

Theorem 6. *The diagonal pentagon of a convex pentagon cannot have two vertex angles that are acute and adjacent.*

Proof. Consider $\triangle ARS$. At most one vertex of the triangle can be obtuse. Thus, at least one of $\angle ARS$ and $\angle ASR$ must be acute, and hence, at least one of their supplements $\angle QRS$ and $\angle TSR$ must be obtuse. The same applies to any pair of adjacent vertex angles of $PQRST$. □

Corollary 7. *The diagonal pentagon of a convex pentagon can have at most two acute vertex angles.*

Proof. Otherwise, two of the acute vertex angles must be adjacent, which contradicts the theorem above. □

Theorem 8. *The five quadrilaterals $ABCQ \ \circlearrowleft$ of a convex pentagon cannot all be cyclic.*

Proof. $A + P = 180^\circ \ \circlearrowleft$, adding these we have $\sum_{\circlearrowleft} A + P = 5 \cdot 180^\circ$. But $\sum_{\circlearrowleft} A = \sum_{\circlearrowleft} P = 3 \cdot 180^\circ$, so $\sum_{\circlearrowleft} A + P = 6 \cdot 180^\circ$, which contradicts the earlier statement. □

In fact, if $ABCDE$ is convex non-degenerate, at most 3 of the quadrilaterals $ABCQ$ $\circlearrowleft$ can be cyclic. Suppose four are cyclic (all but $DEAT$). Then we have

$$A + B + C + D + P + Q + R + S = 4 \cdot 180^\circ$$

Thus, $E + T = 360$. If $ABCDE$ is convex, then $PQRST$ must be convex, and it follows that $E = T = 180^\circ$, which means $ABCDE$ must have all vertices in a straight line (which implies $PQRST$ have all vertices in the same line), which contradicts the requirement that $ABCDE$ is non-degenerate.

The following figure suggests that it is possible for 3 such quads to be cyclic.

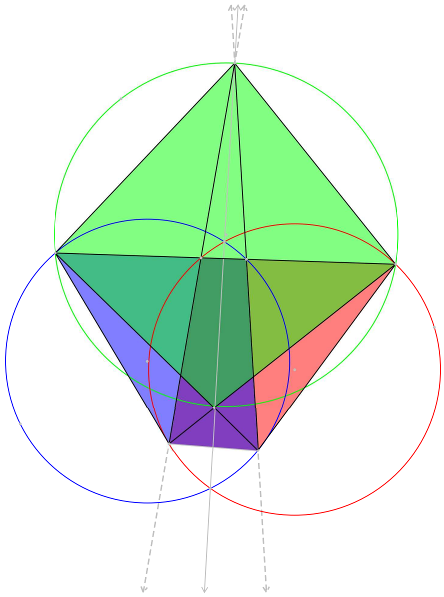

Figure 4: Three cyclic quadrilaterals in a pentagon.

Definition 3. *Given a pentagon $ABBCDE$, the medial pentagon $JKLMN$ is the pentagon such that J bisects CD $\circlearrowleft$.*

Theorem 9. *If $ABCDE$ is a convex pentagon with medial pentagon $JKLMN$, then*

$$\frac{\mathcal{A}\,(JKLMN)}{\mathcal{A}\,(ABCDE)} \in \left(\frac{1}{2}, \frac{3}{4}\right)$$

Theorem 10 (Similarity of Pentagons)**.** *Two pentagons $ABCDE$ and $A'B'C'D'E'$ are similar when any one of the minimum requirements in the table below are satisfied. The number indicates the number of pairs of corresponding vertices that should be equal, and the number of pairs of corresponding sides and diagonals that should be proportional.*

Vertices	*Sides*	*Diagonals*
4	*3 adjacent*	*0*
4	*0*	*3 adjacent*

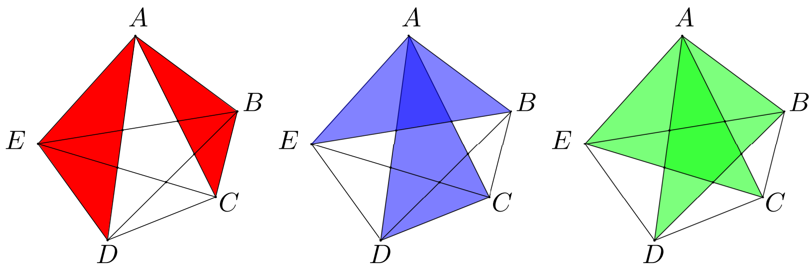

Figure 5: The triangles of Monge's formula

3 General Pentagons

3.1 Monge, Gauss, Ptolemy

Theorem 11 (Monge's Formula [9]). *If $ABCDE$ is a convex pentagon, then*

$$\mathcal{A}\,(ABC)\,\mathcal{A}\,(ADE) + \mathcal{A}\,(ABE)\,\mathcal{A}\,(ACD) = \mathcal{A}\,(ABD)\,\mathcal{A}\,(ACE) \quad \circlearrowleft$$

Proof. In the derivation what follows, we make use of the trigonometric identity

$$\sin\alpha\sin\gamma + \sin\beta\sin(\alpha+\beta+\gamma) = \sin(\alpha+\beta)\sin(\beta+\gamma).$$

We prove Monge's formula for the triangles with vertex A.

$$\begin{aligned}
&\mathcal{A}\,(ABC)\,\mathcal{A}\,(ADE) + \mathcal{A}\,(ABE)\,\mathcal{A}\,(ACD)\\
&= \frac{AB\cdot AC\sin A_3}{2}\,\frac{AD\cdot AE\sin A_1}{2} + \frac{AB\cdot AE\sin A}{2}\,\frac{AC\cdot AD\sin A_2}{2}\\
&= \frac{AB\cdot AC\cdot AD\cdot AE}{4}\,(\sin A_3\sin A_1 + \sin A\sin A_2)\\
&= \frac{AB\cdot AC\cdot AD\cdot AE}{4}\,(\sin(A_2+A_3)\sin(A_1+A_2))\\
&= \frac{AB\cdot AD\sin(A_2+A_3)}{2}\cdot\frac{AC\cdot AE\sin(A_1+A_2)}{2}\\
&= \mathcal{A}\,(ABD)\,\mathcal{A}\,(ACE)
\end{aligned}$$

□

Theorem 12 (Monge's Formula Vector form). *Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ be any four vectors. Then*

$$(\boldsymbol{a}\circ\boldsymbol{b})(\boldsymbol{c}\circ\boldsymbol{d}) + (\boldsymbol{a}\circ\boldsymbol{d})(\boldsymbol{b}\circ\boldsymbol{c}) = (\boldsymbol{a}\circ\boldsymbol{c})(\boldsymbol{b}\circ\boldsymbol{d}).$$

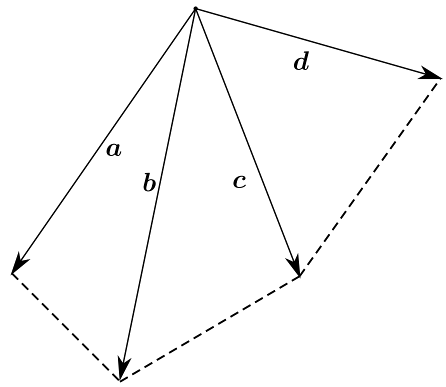

Figure 6: Monge's Formula Vector Form

Proof.

$$
\begin{aligned}
&(\boldsymbol{a} \circ \boldsymbol{b})(\boldsymbol{c} \circ \boldsymbol{d}) + (\boldsymbol{a} \circ \boldsymbol{d})(\boldsymbol{b} \circ \boldsymbol{c}) \\
&\qquad = (a_x b_y - a_y b_x)(c_x d_y - c_y d_x) + (a_x d_y - a_y d_x)(b_x c_y - b_y c_x) \\
&\qquad = a_x b_y c_x d_y - a_x b_y c_y d_x - a_y b_x c_x d_y + a_y b_x c_y d_x \\
&\qquad\qquad + a_x b_x c_y d_y - a_x b_y c_x d_y - a_y b_x c_y d_x + a_y b_y c_x d_x \\
&\qquad = a_x b_x c_y d_y - a_x b_y c_y d_x - a_y b_x c_x d_y + a_y b_y c_x d_x \\
&\qquad = (a_x c_y - a_y c_x)(b_x d_y - b_y d_x) \\
&\qquad = (\boldsymbol{a} \circ \boldsymbol{c})(\boldsymbol{b} \circ \boldsymbol{d})
\end{aligned}
$$

□

The relation to the geometric version of the formula should be clear when you notice that the area of the triangle between two vectors is given by $\frac{1}{2}\boldsymbol{a} \circ \boldsymbol{b} = \frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$, where θ is the anti-clockwise angle between the two vectors.

Theorem 13 (Gauss's Formula)**.** *$ABCDE$ is a convex pentagon. Let*

$$
\begin{aligned}
c_1 &= \sum_{\circlearrowleft} \mathcal{A}(ABC) \\
c_2 &= \sum_{\circlearrowleft} \mathcal{A}(ABC)\,\mathcal{A}(BCD)
\end{aligned}
$$

Then the the area $K = \mathcal{A}(ABCDE)$ of the pentagon is given by the solution of

$$K^2 - c_1 K + c_2 = 0$$

Proof. The proof follows from Monge's formula if we make the following substitutions:

$$
\begin{aligned}
\mathcal{A}(ACD) &= K - \mathcal{A}(ABC) - \mathcal{A}(DEA) \\
\mathcal{A}(ABD) &= K - \mathcal{A}(BCD) - \mathcal{A}(DEA) \\
\mathcal{A}(ACE) &= K - \mathcal{A}(ABC) - \mathcal{A}(CDE)
\end{aligned}
$$

□

For a pentagon with all vertex triangles of equal area $k = \mathcal{A}(\triangle ABC)\ \circlearrowleft$,

$$c_1 = 5k$$
$$c_2 = 5k^2,$$

thus

$$K = \frac{5k + \sqrt{25k^2 + 20k^2}}{2} = \frac{(5+\sqrt{5})k}{2} = \sqrt{5}\phi k$$

Theorem 14 (Ptolomy's Formula)**.** *Let R_{ABC} be the radius of the circumcircle of triangle ABC. Then, for any pentagon $ABCDE$*

$$\frac{BC}{R_{ABC}} \cdot \frac{DE}{R_{ADE}} + \frac{BE}{R_{ABE}} \cdot \frac{CD}{R_{ACD}} = \frac{BD}{R_{ABD}} \cdot \frac{CE}{R_{ACE}} \quad \circlearrowleft$$

Proof. This follows directly from Monge's formula by using $\mathcal{A}(XYZ) = \frac{xyz}{R_{XYZ}}$ and dividing by $AB \cdot AC \cdot AD \cdot AE$. □

3.2 Cyclic Ratio Products (alla Ceva en Melenaus)

Theorem 15. *If $ABCDE$ is a pentagram, then*

$$\prod_{\circlearrowleft} AR = \prod_{\circlearrowleft} AS.$$

Proof. Using the rule of sines, we have

$$\frac{AR}{\sin \angle ASR} = \frac{AS}{\sin \angle ARS} \quad \circlearrowleft .$$

Multiplying these together, we get

$$\prod_{\circlearrowleft} \frac{AR}{\sin \angle ASR} = \prod_{\circlearrowleft} \frac{AS}{\sin \angle ARS},$$

or equivalently,

$$\prod_{\circlearrowleft} \frac{AR}{\sin \angle ASR} = \prod_{\circlearrowleft} \frac{AS}{\sin \angle BST}.$$

But

$$\angle ASR = \angle BST \quad \circlearrowleft$$

because they are vertically opposite angles, hence

$$\prod_{\circlearrowleft} AR = \prod_{\circlearrowleft} AS$$

□

Definition 4 (Cevian)**.** *A cevian of a pentagon is a line passing through a vertex and intersecting the opposite side of the pentagon.*

Lemma 16. *If AD is a cevian of $\triangle ABC$ with D on BC, then*

$$\frac{CD}{DB} = \frac{AC \sin \angle CAD}{AB \sin \angle BAD}$$

Theorem 17.A (Ceva Pentagon Theorem, Larry Hoehn)**.** *Let the five cevians through a point O in the interior of a pentagon intersect the sides. We label the intersection $J = AO \cap CD$ $\circlearrowleft$. Then*

$$\prod_{\circlearrowleft} \frac{AM}{MB} = 1 \tag{1}$$

Proof. Using the lemma above, we have

$$\prod_{\circlearrowleft} \frac{AM}{MB} = \prod_{\circlearrowleft} \frac{AO \sin \angle AOM}{BO \sin \angle BOM}$$

Or after re-arranging factors on the right

$$\prod_{\circlearrowleft} \frac{AM}{MB} = \prod_{\circlearrowleft} \frac{AO \sin \angle AOM}{AO \sin \angle DOJ}$$

Now vertical opposite angles are equal,

$$\angle AOM = \angle DOJ \quad \circlearrowleft$$

thus

$$\prod_{\circlearrowleft} \frac{AM}{MB} = \prod_{\circlearrowleft} \frac{AO \sin \angle AOM}{AO \sin \angle DOJ} = 1$$

□

Theorem 17.B (Ceva Pentagon Theorem, Converse)**.** *If four cevians are concurrent in O and*

$$\prod_{\circlearrowleft} \frac{AM}{MB} = 1, \tag{2}$$

then all five cevians are concurrent.

We can also use the rule of sines to get the theorem in trigonometric form. The concurrency condition is then:

$$\prod_{\circlearrowleft} \frac{\sin \angle OAB}{\sin \angle OBA} = 1.$$

Theorem 18 (Hoehn's Theorem)**.** *For a convex pentagon $ABCDE$,*

$$\prod_{\circlearrowleft} \frac{AS}{TC} = 1 \tag{3}$$

$$\prod_{\circlearrowleft} \frac{AT}{SC} = 1 \tag{4}$$

$$\frac{AS}{TC} = \frac{\mathcal{A}(ABE)}{\mathcal{A}(ABCE)} \cdot \frac{\mathcal{A}(BCDA)}{\mathcal{A}(BCD)} \quad \circlearrowleft \tag{5}$$

Theorem 19 (Melenaus for Pentagons)**.** *If a line intersects the sides (possibly extended) CD in J $\circlearrowleft$, then*

$$\prod_{\circlearrowleft} \frac{CJ}{JD} = -1$$

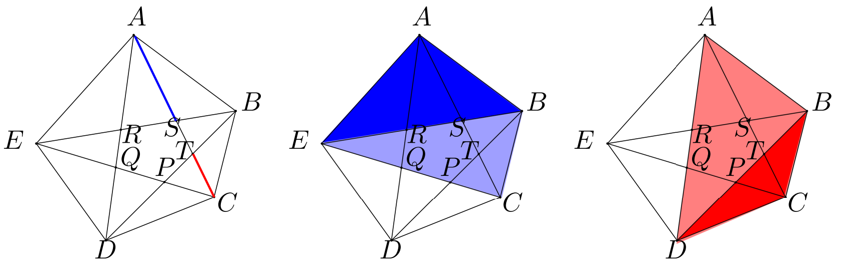

Figure 7: Hoehn's Theorem Mnemonic

3.3 Miquel

The theorems in his section are essentially based on a generalisation of Miquel's theorem for triangles.

Theorem 20.A. *If five circles c_A $\circlearrowleft$ $ABCDE$ intersect in a common point O and five other points J, K, L, M, N, choose any point A on c_A.*

1. *Construct line AM, let it intersect c_B again in B.*
2. *Construct line BN, let it intersect c_C again in C.*
3. *Construct line CJ, let it intersect c_D again in D.*
4. *Construct line DK, let it intersect c_E again in E.*

Then AE goes through L.

Proof. We prove ALE is a straight line.

Join JO $\circlearrowleft$. Then

1. $OLA + OMA = 180$, $OMA + OMB = 180$, thus $OLA = OMB$.
2. $OMB + ONB = 180$, $ONB + ONC = 180$, thus $OLA = ONC$.
3. $ONC + OJC = 180$, $OJC + OJD = 180$, thus $OLA = OJD$.
4. $OJD + OKD = 180$, $OKD + OKE = 180$, thus $OLA = OKE$.
5. $OKE + OLE = 180$.

Thus $OLA + OLE = 180$, hence ALE is a straight line. □

Theorem 20.B. *On any pentagon $ABCDE$, if we mark of M on AB $\circlearrowleft$, and the four circles BMN, CNJ, DJK and EKL all intersect in a point O, then circle ALM also pass through O.*

Theorem 20.C. *On any pentagon $ABCDE$, mark a point M on AB, and choose any point O. Construct $\odot AOM$, let it cut EA in L. Construct $\odot BOM$, let it cut BC in N. Construct $\odot CON$, let it cut CD in J. Construct $\odot DOJ$, let it cut DE in K.*

Then $\odot EKL$ pass through O.

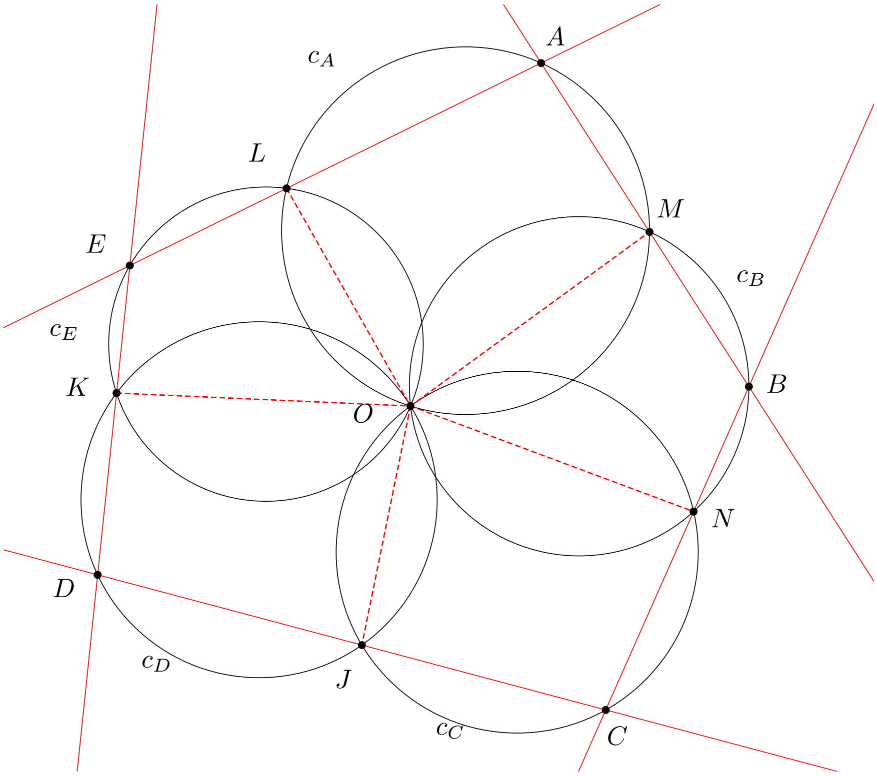

Figure 8: Miquel's Theorem

The last two parts are proved similarly to the first: in each case, we chase angles around the sequence of circles until we can eventually prove $LOKE$ is cyclic.

We have proven that the following arrangement of figures always exist:

Definition 5. *A **Miquel arrangement** is a pentagon $ABCDE$ with five points $J \circlearrowleft$ marked on the sides, and five circles with centers $O_A \circlearrowleft$ that*

- *all intersect in a common point O, and*
- *each intersect a vertex and the two marked points on adjacent sides of the pentagon.*

The remainder of this section deals with Miquel arrangements. Theorem 20 and those that follow are easily generalised to polygons with any number of sides.

Theorem 21. *If in a Miquel arrangement $O_A \in AO$, then $O_A \in AO \circlearrowleft$.*

Theorem 22. *The area of the pentagon in a Miquel arrangement is maximal when $O_A \in AO$.*

Proof. Let $ABCDE$ and $A'C'D'E'$ be two pentagons in Miquel arrangement with the same five circles, and let $O_A \in OA$. The areas of these two pentagons are given by

$$\mathcal{A}(ABCDE) = \mathcal{A}(JKLMN) + \sum_{\circlearrowleft} \mathcal{A}(ALM) \tag{6}$$
$$\mathcal{A}(A'B'C'D'E') = \mathcal{A}(JKLMN) + \sum_{\circlearrowleft} \mathcal{A}(A'LM) \tag{7}$$

□

Theorem 23. *The centres O_A $\circlearrowleft$ of five circles in a Miquel arrangement form a pentagon, and $O_AO_BO_CO_DO_E \sim ABCDE$.*

Corollary 24. *If the five circles c_A have equal radius, then the pentagon $ABCDE$ is cyclic.*

Proof. Since the five circles have equal radius and they all intersect in O, it follows that their centers are concyclic, that is the pentagon $O_AO_BO_CO_DO_E$ is concyclic, and since it is similar to $ABCDE$, the pentagon $ABCDE$ too must be concyclic. □

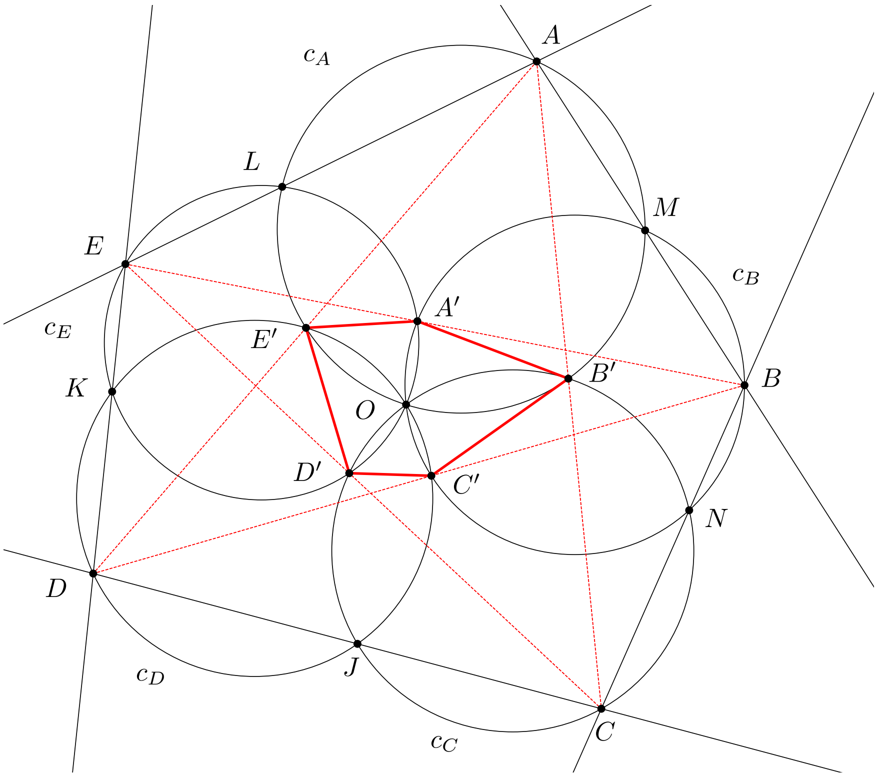

Figure 9: Theorem 25.

Theorem 25. *In the Miquel arrangement, pairs of circles intersect in points A' (in addition to the point O). The pentagon $A'B'C'D'E'$ is inscribed in the diagonal pentagon.*

Proof. This follows easily from Miquel's theorem for triangles. Circles O_A, O_B, and O_C intersect in a common point O. Since the intersection of $\odot O_A$

and $\odot O_B$ (other than O) lies on AB, and the intersection of $\odot O_B$ and $\odot O_C$ (other than O) lies on BC, the intersection of $\odot O_A$ and $\odot O_C$ must lie on AC. Similarly, $\odot O_A \cap \odot O_C$ (not O) $\in AC$ $\circlearrowleft$. □

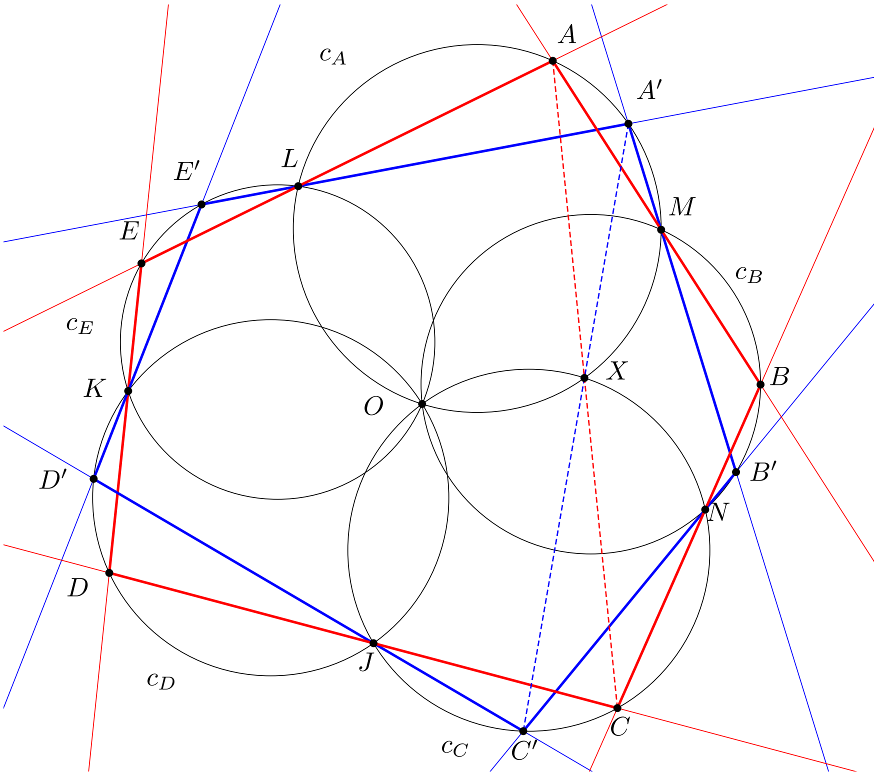

Figure 10: Theorem 26

Theorem 26. *If $ABCDE$ and $A'B'C'D'E$ are two pentagons constructed on the same five circles intersecting in a common point O, (with different initials points A and A'), then $ABCDE \sim A'B'C'D'E'$.*

Proof. By Theorem 25 AC, $A'C'$, c_A and c_C all have a common point which we label X. From this, it follows that $\angle XAM = \angle XA'M$, since these are angles in $\odot c_A$ suspended by chord XM. Similarly, $\angle XCN = \angle XC'N$. Finally, $\angle MBN = \angle MB'N$. Thus $ABC \sim A'B'C'$. We can show similarly $ABC \sim A'B'C'$ $\circlearrowleft$. And thus, by Theorem 10, we have $ABCDE \sim A'B'C'D'E'$. □

3.4 Conics

Theorem 27. *For a pentagon, we can find a unique conics that passes through all the vertices of the pentagon.*

From this, it should be clear that there exists a projective transformation from any pentagon to a cyclic pentagon.

The following gives a method of constructing the circumconic of pentagon $ABCDE$ [6]:

(1) $S_2 := AB \cap DE$

(2) Let g be any line through S_2.

(3) $S_1 := g \cap BC$.

(4) $S_3 := g \cap CD$.

(5) $X := S_1A \cap S_3E$.

Then X lies on the circumconic of $ABCDE$.

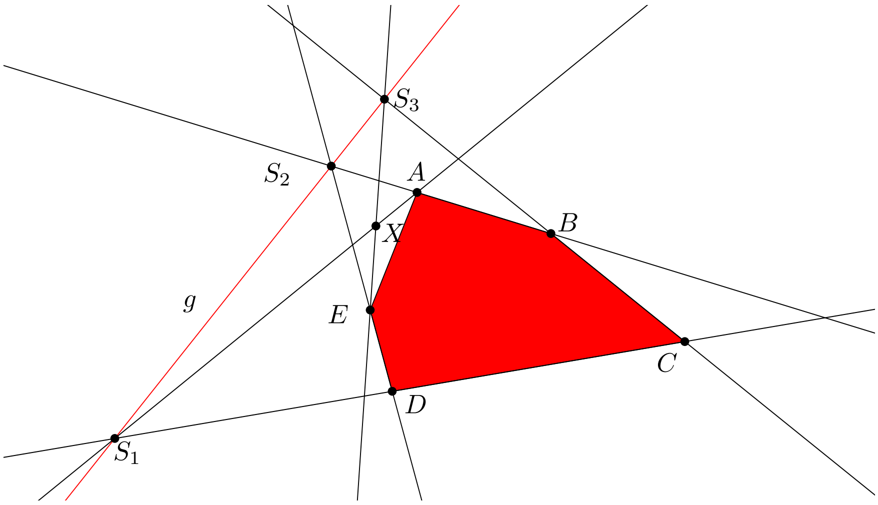

Figure 11: Construction of a conic

Theorem 28. *For a pentagon, we can find a unique conic that is tangent to the five sides of the pentagon.*

3.5 Complete Pentagons

In a complete quadrilateral, we can join three pairs of non-adjacent vertices to form three diagonals. The midpoints of the three diagonals are lie on a common line, the **Newton-Gauss line** of the quadrilateral.

Four triangles are formed if we take three sides at a time. The circumcircles of these triangles share a common point, the **Cliffort point** of the quadrilateral. The centers of these circles are concyclic; the common circle is called the **Morley circle** of the quadrilateral. The Cliffort point also lies on the Morley circle.

Theorem 29 (Grunert's Theorem [3])**.** *Let $A' = BC \cap DE$ ⟳. The, let J bisect AA' ⟳, and J' bisect EB ⟳. Then the lines JJ' ⟳ are concurrent, provided that they all intersect and the dual pentagon $ACEBD$ has non-zero area.*

Proof. □

The point of concurrency is called the **Grunert point** of the pentagon. This theorem is implicit in a lemma by Newton [2]. The lines JJ' ⟳ are the Newton-Gauss lines of the quadrilaterals $EABA'$.

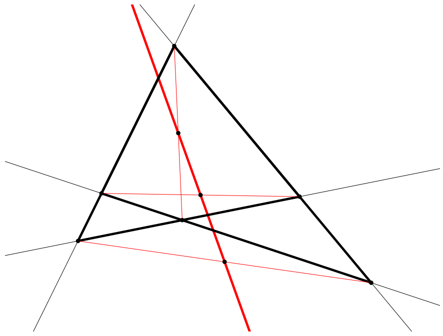

Figure 12: Newton-Gauss Line

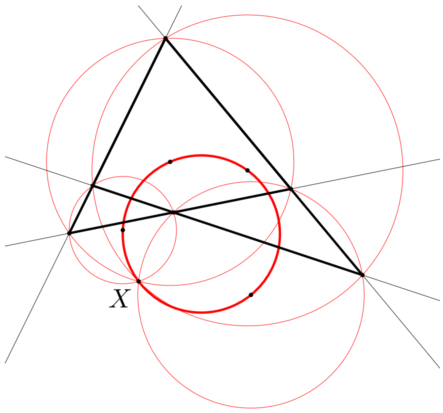

Figure 13: Morley Circle

Theorem 30 (Newton's Theorem [2])**.** *Let $M = AB \cap CD$ $\circlearrowleft$. Then the Newton-Gauss line of the quadrilaterals $AMDE$ $\circlearrowleft$ are concurrent in the center of the inscribed conic of the pentagon.*

Theorem 31 (Morley Circle)**.** *The centers of the five Morley circles of the quadrilaterals, formed by four sides at a time, is concyclic.*

Theorem 32 (Miquel's Pentagram Theorem)**.** *Let $ABCDE$ be a pentagram that self-intersects $P = BD \cap EC$ $\circlearrowleft$. Then $\odot ARS \cap \odot BST = \{S, S'\}$ $\circlearrowleft$, and $P'Q'R'S'T'$ is cyclic.*

Theorem 33 (Clifford Circle)**.** *Let $M = AB \cap CD$ $\circlearrowleft$. Then the Clifford points F_E of $AMDE$ $\circlearrowleft$ are concyclic.*

The common circle is called the **Clifford circle** of the pentagon.

3.6 The Centroid Theorem

In this section I present a theorem on general polygons by Mammana, Micale and Pennisi [7], with some applications to pentagons.

Definition 6 (Centroid)**.** *Let $\boldsymbol{x}_i$ be the position vectors of a set of n points P_i with respect to a fixed point P. The centroid the set of points P_i is the*

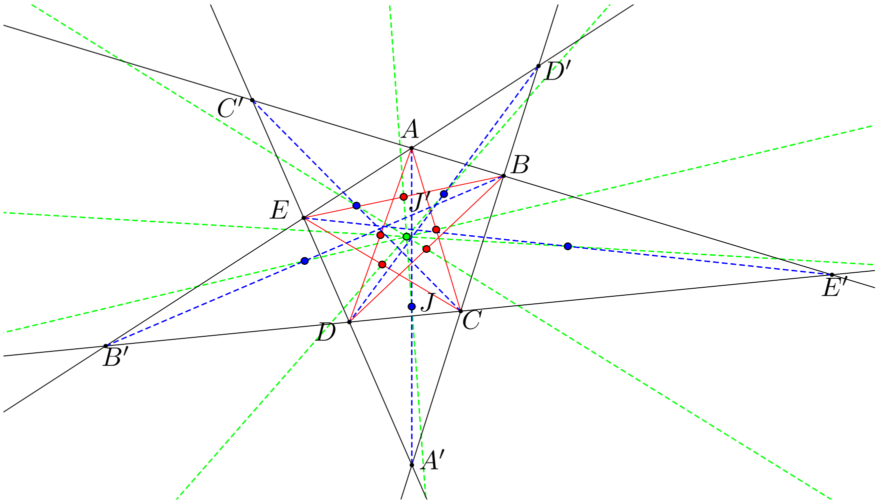

Figure 14: Grunert Point

point with position vector with respect to P

$$\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i.$$

We denote this point by $\mathcal{C}\left(P_1P_2\cdots P_n\right)$.

The centroid of a polygon is the centroid of the vertex points.

It should be clear that the centroid does not depend on P or the order of the points, so that the concept of the centroid of a polygon is unambiguous.

Theorem 34. *If we partition the vertex points of polygon with* n *vertices into two disjoint sets with* k *and* $n-k$ *points each, the line through the centroids (*A *and* B*) of the two sets lies on the centroid* C *of the polygon, and*

$$\frac{AC}{CB} = \frac{n-k}{k}$$

Proof. Let the k points of the one set be $A_1\cdots A_k$, and the points of the other set be $B_1\cdots B_{n-k}$. Choose a fixed point P, and let $\boldsymbol{a}_i$ be position vectors of A_i with respect to P, and $\boldsymbol{b}_i$ be position vectors of B_i with respect to P. The centroid of the polygon is the point C with position vector

$$\boldsymbol{c} = \frac{1}{n}\left[\sum_{i=1}^{k}\boldsymbol{a}_i + \sum_{i=1}^{n-k}\boldsymbol{b}_i\right]$$

The centroids of the two partitions have position vectors

$$\boldsymbol{a} = \frac{1}{k}\sum_{i=1}^{k}\boldsymbol{a}_i$$

$$\boldsymbol{b} = \frac{1}{n-k}\sum_{i=1}^{n-k}\boldsymbol{b}_i$$

So

$$\begin{aligned} n\boldsymbol{c} &= k\boldsymbol{a} + (n-k)\boldsymbol{b} \\ \Rightarrow k\boldsymbol{c} - k\boldsymbol{a} &= (n-k)\boldsymbol{b} - (n-k)\boldsymbol{c} \\ \Rightarrow \boldsymbol{c} - \boldsymbol{a} &= \frac{n-k}{k}(\boldsymbol{b} - \boldsymbol{c}) \end{aligned}$$

This means that the lines AC and BA are the same line (thus C lies on AB), and that

$$\frac{AC}{BA} = \frac{|\boldsymbol{c} - \boldsymbol{a}|}{|\boldsymbol{b} - \boldsymbol{c}|} = \frac{n-k}{k}$$

□

Theorem 35 (Centroids of Pentagons)**.** *If $ABCDE$ is a pentagon with centroid O, then*

(1) *Let $J = \mathcal{C}(ABC)$ $\circlearrowleft$ and $J' = \mathcal{C}(DC)$ $\circlearrowleft$. Then the lines $\circlearrowleft$ JJ' are concurrent in O.*

(2) *Let $J = \mathcal{C}(ABCD)$ $\circlearrowleft$. Then the lines JD $\circlearrowleft$ are concurrent in O.*

(3) *Let $J = \mathcal{C}(AB)$ $\circlearrowleft$, $J' = \mathcal{C}(BC)$ $\circlearrowleft$, and $K = \mathcal{C}(CDE)$ $\circlearrowleft$, and $K' = \mathcal{C}(ADE)$ $\circlearrowleft$. Then $JJ' \parallel KK'$ $\circlearrowleft$, and $|JJ'|/|KK'| = 3/2$ $\circlearrowleft$.*

(4) *Let $J = \mathcal{C}(ABCD)$ $\circlearrowleft$, $J' = \mathcal{C}(BCDE)$ $\circlearrowleft$. Then $JJ' \parallel EA$ $\circlearrowleft$, and $|JJ'|/|EA| = 1/4$ $\circlearrowleft$.*

We can generalize the theorem as follows. Let f_n be an isometry invariant function of the vertices of a n-gon.

Then the f_n-centroid of a point set X_i with position vectors $\boldsymbol{x}_i$ is the point with position vector

$$\bar{\boldsymbol{x}}_{f_n} = \frac{\sum_{i=1}^{n} \boldsymbol{x}_i f_n(X_i)}{\sum_{i=1}^{n} f_n(X_i)}$$

and denoted by $\mathcal{C}_{f_n}(X_1 X_2 \cdots X_n)$. All position vectors are with respect to some fixed point P.

Here are some examples of candidate functions for a pentagon $ABCDE$:

$$\begin{aligned} f_5(A) &= 1 \ \circlearrowleft \\ f_5(A) &= \sin A \ \circlearrowleft \\ f_5(A) &= |CD| \ \circlearrowleft \\ f_5(A) &= \mathcal{A}(ABC) \ \circlearrowleft \end{aligned}$$

Theorem 36 (f_n-Centroids)**.** *If we partition the vertex points of polygon with n vertices into two disjoint sets with k and $n-k$ points each, the line through the f_n-centroids (A and B) of the two sets lies on the f-centroid C of the polygon, and*

$$\frac{AC}{CB} = \frac{\sum_{i=1}^{n-k} f_n(B_i)}{\sum_{i=1}^{k} f_n(A_i)}$$

The proof is exactly as the centroid theorem above.

If we choose $f_n(X_i) = 1$, we get the centroid theorem above.

As an example on pentagons, choose $f_5(A) = |CD| \;\circlearrowleft$, and let us partition into vertex triangles and opposite sides, so that we have for each vertex the points

$$\mathcal{C}_{f_5}(ABE) = \frac{CD \cdot \boldsymbol{a} + DE \cdot \boldsymbol{b} + BC \cdot \boldsymbol{e}}{CD + DE + BC}$$
$$\mathcal{C}_{f_5}(CD) = \frac{EA \cdot \boldsymbol{c} + AB \cdot \boldsymbol{d}}{EA + AB}$$

Then the lines $\mathcal{C}_{f_5}(ABE)\mathcal{C}_{f_5}(CD) \;\circlearrowleft$ are all concurrent.

4 Special Pentagons

4.1 Cyclic Pentagons

The following theorem is one of many $4 \to 5$-type theorems:

Theorem 37. *If four perpendicular bisectors of the sides of a pentagon are concurrent, then all five perpendicular bisectors are concurrent.*

Proof. Suppose the perpendicular bisectors of AB, BC, CD, and DE are concurrent in O, and suppose they intersect the sides in M, N, J and K. Then $\triangle AOM \equiv \triangle BOM$ (SAS), so $AO = BO$. With similar arguments we have $AO = BO = CO = DO = EO$, so all points A, B, C, D, E lie on a common circle with centre O. Thus EA is a chord, and its perpendicular bisector passes through the center O. $\square$

We begin with a few obvious properties of cyclic pentagons:

Theorem 38. *If $ABCDE$ is a cyclic pentagon, then*

(1) *the perpendicular bisectors of the sides are concurrent,*

If $ABCDE$ is also convex, then

(1) $AC \cdot BD = AB \cdot CD + AD \cdot BC \;\circlearrowleft$,

(2) $A_1 = B_2 = C_3 \;\circlearrowleft$

Theorem 39 (The rule of sines)**.** *If $ABCDE$ is cyclic, then*

$$\frac{\sin A}{EB} = \frac{\sin B}{AC} \;\circlearrowleft .$$

Proof. Using rules of sines for triangle ABE, we have $\frac{\sin A}{EB} = \frac{\sin E_3}{AB}$. Because $ABCDE$ is cyclic, $E_3 = C_1$, thus $\frac{\sin E_3}{AB} = \frac{\sin C_1}{AB}$, and using rules of sines for ABC we have $\frac{\sin C_1}{AB} = \frac{\sin B}{AC}$. $\square$

It should be clear that the theorem is easily extended to any cyclic polygon.

Theorem 40.

(1) *If a convex pentagon is cyclic, and its diagonal pentagon are both cyclic, the pentagon is regular.*

(2) *If a convex pentagon and its extended pentagram are both cyclic, then the pentagon is regular.*

Theorem 41. *If $ABCDE$ is cyclic, and J is the bimedian of quadrangle $BCDE$ $\circlearrowleft$, then the pentagon $JKLMN$ $ABCDE$, and $JK \parallel AB$.*

4.2 Tangent Pentagons

Definition 7. *Tangent pentagons are pentagons that can circumscribe a circle.*

Theorem 42. *If four angular bisectors of a pentagon are concurrent, then all five angular bisectors are concurrent.*

Theorem 43. *The angular bisectors of a tangent pentagon are concurrent.*

Proof. This follows from the fact that the line bisecting the angle of two tangents from a common point at that point passes through the circle. □

Theorem 44. *If four angular bisectors are concurrent, then all five angular bisectors are concurrent.*

Preposition 45. *To construct a tangent pentagon with three sides AB, BC, CD given.*

(1) Construct the angular bisector b through B.

(2) Construct the angular bisector c through D.

(3) Let $O = b \cap c$.

(4) Reflect the line AB through AO, call it x.

(5) Reflect the line CD through DO, call it y.

(6) Let $E = x \cap y$.

4.3 Orthocentric Pentagons

Definition 8 (Orthocentric Pentagon)**.** *An ortho-centric pentagon is a pentagon whose altitudes are concurrent.*

Theorem 46. *If four altitudes of a pentagon are concurrent, then all five altitudes are concurrent.*

Preposition 47 (Constructing a orthocentric pentagon given five concurrent lines that coincide with the five altitudes, and a vertex)**.** *Let the five lines be a, b, c, d and e, concurrent in O.Let the vertex be A.*

(1) Draw $AB \perp d$, with B on b.

(2) Draw $BC \perp e$, with C on c.

(3) Draw $CD \perp a$, with D on d.

(4) Draw $DE \perp b$, with E on e.

(5) Connect EA.

Then $ABCDE$ is a orthocentric pentagon.

Proof. All we have to do is prove c is an altitude. Since a, b, d, e are all altitudes concurrent in O by construction, it follows that the fifth altitude must also pass through O. But c already passes through O, so c must be the altitude. □

Preposition 48 (Constructing a orthocentric pentagon with three sides given.)**.** *Let A, B, C, and D be the four sides given.*

(1) Construct $a \perp CD$.

(2) Construct $d \perp AB$.

(3) Let $O = a \cap d$.

(4) Construct $e \perp BC$.

(5) Construct $DE \perp b$, with E on e.

Then $ABCDE$ is orthocentric. The fifth altitude is the line $c = CO$.

Proof. a, b, d, and e are all altitudes of the pentagon by construction, and concurrent in O. The fifth altitude c must then also be concurrent through O, but CO is already a line through C and O, and hence that line must be c, the fifth altitude. □

Theorem 49. *If $ABCDE$ is orthocentric with centre O and extended pentagram $JKLMN$, than $JO \perp BE$ $\circlearrowleft$.*

Proof. BO and EO are altitudes of $\triangle JBE$. They intersect in O, so the third altitude of $\triangle JBE$ must also pass through O. Thus, JO is an altitude, and hence $JO \perp BE$. With the same argument we can prove all $JO \perp BE$ $\circlearrowleft$. □

4.4 Mediocentric Pentagons

Definition 9 (Mediocentric)**.** *A pentagon is mediocentric when its five medians are concurrent.*

Theorem 50. *If four medians of a pentagon are concurrent, then all five medians are concurrent.*

Proof. The proof is direct using Ceva's Theorem, since for medians $AM = MB$ $\circlearrowleft$. □

An alternative vector-algebraic proof (for all polygons with odd vertex counts) is given in [8].

Preposition 51 (To construct a mediocentric pentagon given its five medians (concurrent) and a vertex point on one of the medians)**.** *Label the point of concurrence O, and the given vertex A, and the line that it is on a (this is also line AO). Label the other lines b, c, d and e such that d lies between a and b $\circlearrowleft$, as shown in the figure. Then*

(1) *Construct $AM \parallel b$, with M on d. Construct $MB \parallel a$ with B on b.*

(2) *Construct $BN \parallel c$, with N on e. Construct $NC \parallel b$ with C on c.*

(3) *Construct $CJ \parallel d$, with J on a. Construct $JD \parallel c$ with D on d.*

(4) *Construct $DK \parallel e$, with K on b. Construct $KE \parallel d$ with E on e.*

Now connect AB, BC, CD, DE and EA. $ABCDE$ is a mediocentric pentagon.

Proof. $AOBM$ is a parallelogram, so OM bisects AB, thus, d connects a vertex D with the midpoint of the opposite site AB, so d is a median.

Similarly, by considering paralelellograms $BOCN$, $CODK$ and $DOEL$, we find that e, a and b are medians. These medians are concurrent (given), so it follows that the fifth median will also be concurrent. Hence, this pentagon is mediocentric. Since only one line can pass through C and O, it must be the median, so c is in fact the fifth median. □

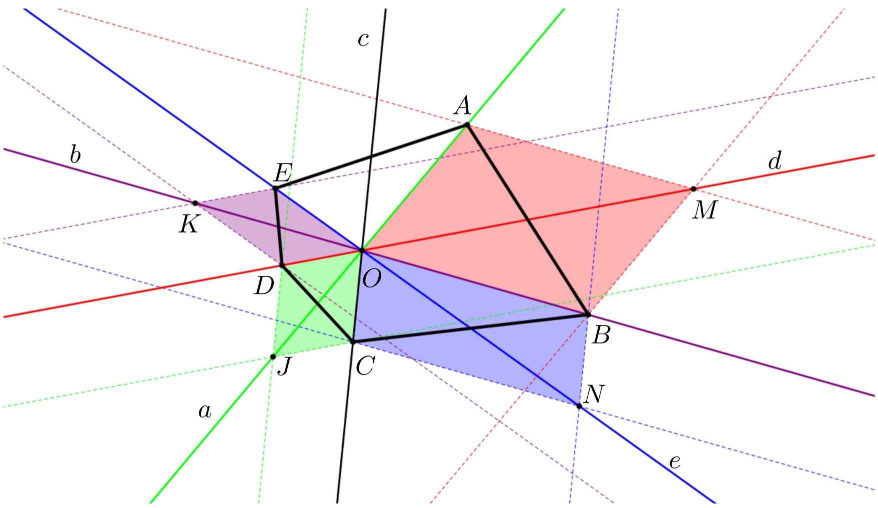

Figure 15: Construction of a mediocentric pentagon.

The same construction works if we label the lines differently. The next figure shows, for example, an arrangement that yields a mediocentric pentagram.

The ideas in the above construction allows us to construct a mediocentric pentagon given three sides. Three sides ($ABCD$) determine the five medians as follows:

(1) Join A with midpoint of CD to find median a.

(2) Join D with midpoint of AB to find median d.

(3) Let $O = a \cap d$.

(4) OB is median b

(5) OC is median c

(6) Join midpoint of BC to O to find median e.

To construct E so that the pentagon is mediocentric, construct $DK \parallel e$, with K on b, and finally construct $KE \parallel d$ with E on e. To complete the pentagon, join DE and AE.

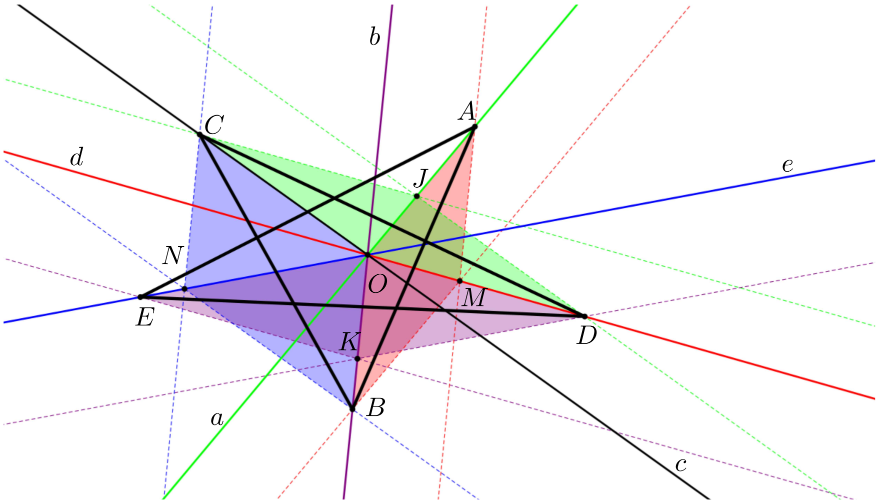

Figure 16: Construction of a mediocentric pentagram.

Theorem 52. *If $ABCDE$ is a mediocentric pentagon with centroid X, with medians cuting opposite sides in J $\circlearrowleft$, then*

$$\prod_{\circlearrowleft} \frac{AX}{XJ} \leq (\sqrt{5}-1)^5$$

Theorem 63 give a condition for equality to hold.

4.5 Paradiagonal Pentagons

This section deals with a class of pentagons that has been studied under various names, starting with different definitions. I use the term paradiagonal for the principle term, because, to me, the definition that inspired the name seems to be the most natural, especially since it alludes to the quadrangle analogue *parallelogram.*

Definition 10 (Paradiagonal Pentagon)**.** *A paradiagonal pentagon has sides parallel to opposite diagonals, that is,*

$$AB \parallel EC \ \circlearrowleft .$$

Definition 11 (Affine Regular)**.** *A pentagon is affine regular if an affine transformation exists that will transform it into a regular pentagon (or, equivalently, if it is a regular pentagon transformed under a affine transformation).*

Definition 12 (Golden Pentagon)**.** *A pentagon in which diagonals cut each other in the golden ration, that is*

$$\frac{SC}{AS} = \phi$$

Definition 13 (Equal Area Pentagon)**.** *A pentagon whose vertex triangles all have equal area.*

Theorem 53. *The following statements are equivalent:*

(1) *A convex pentagon is paradiagonal.*

(2) *A convex pentagon is affine regular.*

(3) *A convex pentagon is golden.*

(4) *A convex pentagon is equal area.*

Proof. (I ommit the label convex here, all the pentagons here are convex).

First, we proof that a paradiagonal pentagon has the properties required by the other definitions. Then we prove the converse of these to show that a pentagon of any of the other definitions is paradiagonal. These taken together proves the theorem.

A paradiagonal pentagon is affine regular

A paradiagonal pentagon is golden

A paradiagonal pentagon is equal area. We have $\mathcal{A}(ABE) = \mathcal{A}(ABC)$, since these triangles lie on a common base AB between parallel lines $AB \parallel EC$. Similarly, we can show that other adjacent pairs of vertex triangles have equal area. It follows that all vertex triangles have equal area.

An affine regular pentagon is paradiagonal. In a regular pentagon, it is easy to see that all vertex triangles are congruent (SAS), hence all diagonals are equal in length. From this it also follows that all edge triangles are congruent. Triangles BCE and EDB are congruent, and hence have equals area. But they also share a base (BE), so $BE||CD$. Similarly, we can prove for other edge and diagonal pairrs, so $BE||CD \circlearrowleft$.

Affine transformations preserve parallelism, so $BE||CD \circlearrowleft$ for a regular pentagon transformed with any affine transformation; thus all affine regular pentagons are paradiagonal.

A golden pentagon is paradiagonal. Through S, construct $SX \parallel AD$ with X on DC. From this, $\frac{AS}{SC} = \frac{DX}{XC}$. But $\frac{DQ}{QA} = \frac{AS}{SC}$, thus $\frac{DX}{XC} = \frac{DQ}{QA}$, thus $XQ \parallel CA$. Hence □

Theorem 54. *If, in a pentagon, four diagonals are parallel to opposite sides, then all five diagonals are parallel to opposite sides.*

Proof. Suppose $BC \parallel AD$, $CD \parallel BE$, $CD \parallel BE$, $CD \parallel BE$. Then we have $\mathcal{A}(ABC) = \mathcal{A}(BCD)$, $\mathcal{A}(BCD) = \mathcal{A}(CDE)$, $\mathcal{A}(CDE) = \mathcal{A}(DEA)$, $\mathcal{A}(DEA) = \mathcal{A}(EAB)$. So $\mathcal{A}(ABC) = \mathcal{A}(EAB)$, which means $AB \parallel EC$. □

We may think that if four vertex triangles have equal area, all four vertex triangles have equal area. However, this is not the case. Here is a way to construct a counter example:

(1) Choose three points in general position, A, B, and C.

(2) Construct line d parallel to AB through C.

(3) Mark a moveable point E on d.

(4) Construct a line e parallel to BC through A.

(5) Construct a line parallel to BE through C, and let it intersect e in D.

From triangles with the same bases and lying between the same sets of parallel lines, it follows that four vertex triangles have equal area:

$$\mathcal{A}(\triangle EAB) = \mathcal{A}(\triangle ABC) = \mathcal{A}(\triangle BCD) = \mathcal{A}(\triangle CDE),$$

regardless of where we position E on d (to keep things simple, let's always keep it on the opposite side of e from C). It is easy to see that we can make the area of $\triangle DEA$ anything we want, from zero to infinity, without changing the areas of any of the other vertex triangles.

This construction also illustrates that we cannot weaken the theorem above to three diagonals parallel to opposite sides, as three sides parallel to opposite diagonals do not imply the other sides are parallel to opposite diagonals.

A similar construction shows that if the four edge triangles have equal area, it does not follow that all five edge triagnles have the same area:

(1) Choose three points in general position, A, B, and D.

(2) Construct line c parallel to DB through A.

(3) Mark a movable point E on c.

(4) Construct a line e parallel to AD through B.

(5) Construct a line parallel to BE, and let it intersect e in C.

Using triangles on equal bases between the same parallel lines, we have

$$\mathcal{A}(\triangle ABD) = \mathcal{A}(\triangle BCE) = \mathcal{A}(\triangle CDA) = \mathcal{A}(\triangle DEB),$$

but by moving E on c we can make $\triangle EAC$ have any area we want, from zero to infinity.

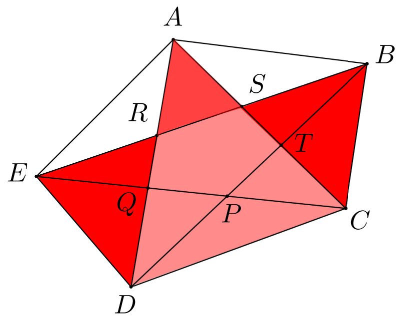

Figure 17: Similar triangles of a paradiagonal pentagon

Theorem 55. *If $ABCDE$ is a paradiagonal pentagon, then*

(1) $ERD \equiv SBC \sim ARS \sim ADC$ ↺

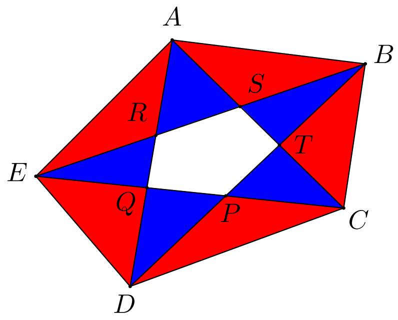

Figure 18: Triangles with equal area

(2) $\mathcal{A}(AER) = \mathcal{A}(ASB)$ $\circlearrowleft$

(3) $\mathcal{A}(ABC) = \mathcal{A}(BCD)$ $\circlearrowleft$

(4) $\mathcal{A}(ARS) = \mathcal{A}(BST)$ $\circlearrowleft$.

(5) $\mathcal{A}(ACD) = \mathcal{A}(BDE)$ $\circlearrowleft$.

Proof. □

For a discussion on equal-area polygons (polygons that have vertex triangles $P_1P_2P_3$ $\circlearrowleft$ $P_1 \cdots P_n$ of equal area), see [5].

Theorem 56 (Golden ratio)**.** *A paradiagonal pentagon satisfies*

$$\frac{SC}{AS} = \frac{AC}{SC} = \frac{AC}{ED} = \phi \ \circlearrowleft .$$

Proof. This ratio theorem is well-known for regular pentagons. Since affine maps preserve ratio's, the golden ratio relationship must also be satisfied for affine regular pentagons, i.e. it must hold for paradiagonal pentagons. □

Because of this theorem, paradiagonal pentagons are also called *golden pentagons*. It also gives us a simple method to construct a paradiogonal pentagon given three adjacent vertices.

Preposition 57 (To construct a paradiagonal pentagon given three adjacent vertices)**.** *Three adjacent vertices, A, B, and C are given.*

(1) Connect AB, BC.

(2) Complete parallelogram $ABCQ$.

(3) On CQ, mark P such that $QP/PC = PC/QC$. One way to do that is as follows [**?**]:

  (a) Draw $QX \perp QC$, with $QX = QC/2$.

  (b) Join XC.

  (c) Mark Y on XC such that $XY = XQ$.

(d) Mark P on CQ such that $CP = CX$.

(4) Let BP extended meet AQ extended in D.

(5) Draw $AE \parallel BD$, with E on CQ extended.

(6) Join CD and DE.

The pentagon $ABCDE$ is a paradiagonal pentagon.

Proof. We need to prove that $CD \parallel BE$ and $AC \parallel ED$. We do this by using equal area arguments. $CP/PQ = CP/CQ = \phi$, but $CQ = BA$ (parallelogram $ABCQ$), thus CP/BA. □

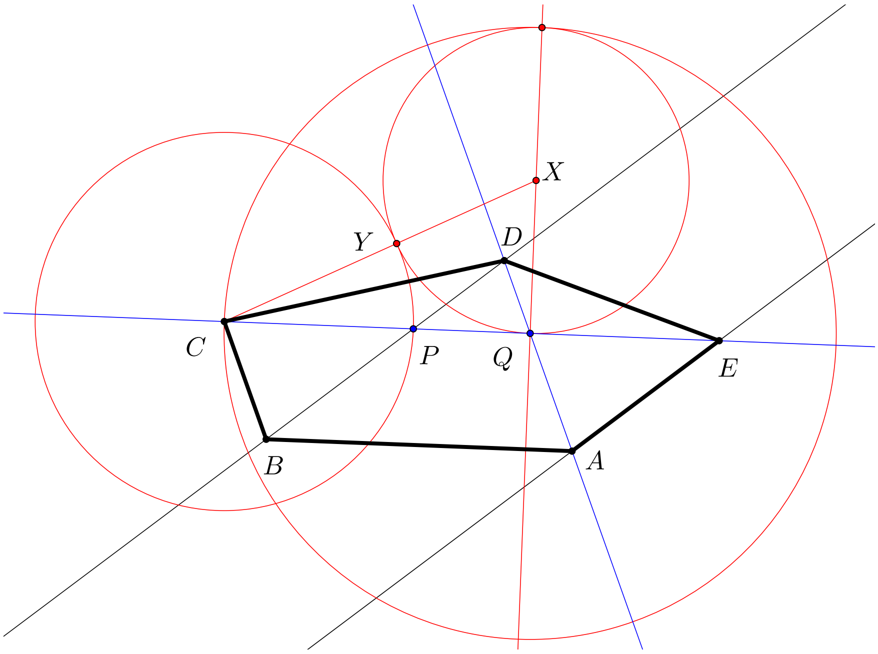

Figure 19: Construction of a paradiagonal pentagon, given three vertices A, B, and C.

Theorem 58. *The diagonal pentagon $PQRST$ of a paradiagonal pentagon $ABCDE$ is similar to it:*

$$ABCDE \sim PQRST$$

Theorem 59. *Let $ABCDE$ be a paradiagonal pentagon, then the medial pentagon $JKLMN$ (where J bisect CD $\circlearrowleft$) is similar to the paradiagonal pentagon:*

$$JKLMN \sim PQRST$$

In the more general case, when $ABCDE$ is an arbitrary pentagon, we have $\angle A = \angle J$ $\circlearrowleft$, but this does not imply the figures are similar.

Theorem 60. *The lines AP $\circlearrowleft$ of a paradiagonal pentagon are concurrent, and they bisect opposite sides CD $\circlearrowleft$.*

Proof. We show that AP, BQ and DS are concurrent.

Let $X = AB \cap DS$. Then, using Ceva's theorem in $\triangle DAB$, we have

$$\frac{AX}{XB} \cdot \frac{BT}{TD} \cdot \frac{DR}{RA} = 1,$$

so that

$$\frac{AX}{XB} = \frac{TD}{BT} \cdot \frac{RA}{DR}.$$

Thus, using the above and the golden ratio theorem,

$$\begin{aligned}\frac{AX}{XB} \cdot \frac{BP}{PD} \cdot \frac{DQ}{DA} &= \frac{TD}{BT} \cdot \frac{RA}{DR} \cdot \frac{BP}{PD} \cdot \frac{DQ}{DA} \\ &= \phi \cdot \frac{1}{\phi} \cdot \phi \cdot \frac{1}{\phi} \\ &= 1\end{aligned}$$

So the triplet of ratios satisfies Ceva's formula, and hence $DX = DS$, AP and BQ are concurrent. We can repeat the proof for other triplets, and eventually establish that all five lines $AP \circlearrowleft$ are concurrent.

To prove that the lines $AP \circlearrowleft$ bisect opposites sides, let $J = AP \cap CD$.

Now, we have the following relationships:

(1) $\frac{DR}{RD} = \phi$ (Theorem 56)

(2) $\frac{AS}{SC} = 1/\phi$ (Theorem 56)

(3) $\frac{DR}{RA} \cdot \frac{AS}{SC} \cdot \frac{CJ}{JD} = 1$ (Ceva's Theorem, since AJ, DS, CR are concurrent)

Combining these give $\frac{CJ}{JD} = 1$, thus $CJ = JD$. Similar arguments prove $CJ = JD \circlearrowleft$. □

Corollary 61. *If a pentagon is paradiagonal, it is also mediocentric.*

Theorem 62 (Congruence SAS)**.** *if two sides and the enclosed angle of two paradiagonal pentagons are equal, then the two pentagons are congruent.*

Theorem 63.A. *Let the medians cut opposite sides in $J \circlearrowleft$, and let $AJ \circlearrowleft$ all intersect in X. Then*

$$\prod_{\circlearrowleft} \frac{AX}{XJ} = (\sqrt{5} - 1)^5$$

Theorem 63.B. *If $ABCDE$ is a mediocentric pentagon with centroid X, and*

$$\prod_{\circlearrowleft} \frac{AX}{XJ} = (\sqrt{5} - 1)^5,$$

then the pentagon is paradiogonal.

4.6 Equilateral Pentagons

Theorem 64. *The two sub angles on the base of vertex triangles of a pentagon are equal, that is*

$$A_3 = C_1 \quad \circlearrowleft$$

Proof. The proof follows immediately from the fact that $ABC \ \circlearrowleft$ are isosceles triangles, which have equal base angles $A_3 = C_1$. $\square$

Theorem 65 (Ptolomy's Formula for Equilateral Pentagons)**.** *Let R_{ABC} be the radius of the circumcircle of triangle ABC. Then, for any cyclic pentagon $ABCDE$*

$$\frac{1}{R_{ABC}} \cdot \frac{1}{R_{ADE}} + \frac{1}{R_{ABE}} \cdot \frac{1}{R_{ACD}} = \frac{1}{R_{ABD}} \cdot \frac{1}{R_{ACE}} \quad \circlearrowleft$$

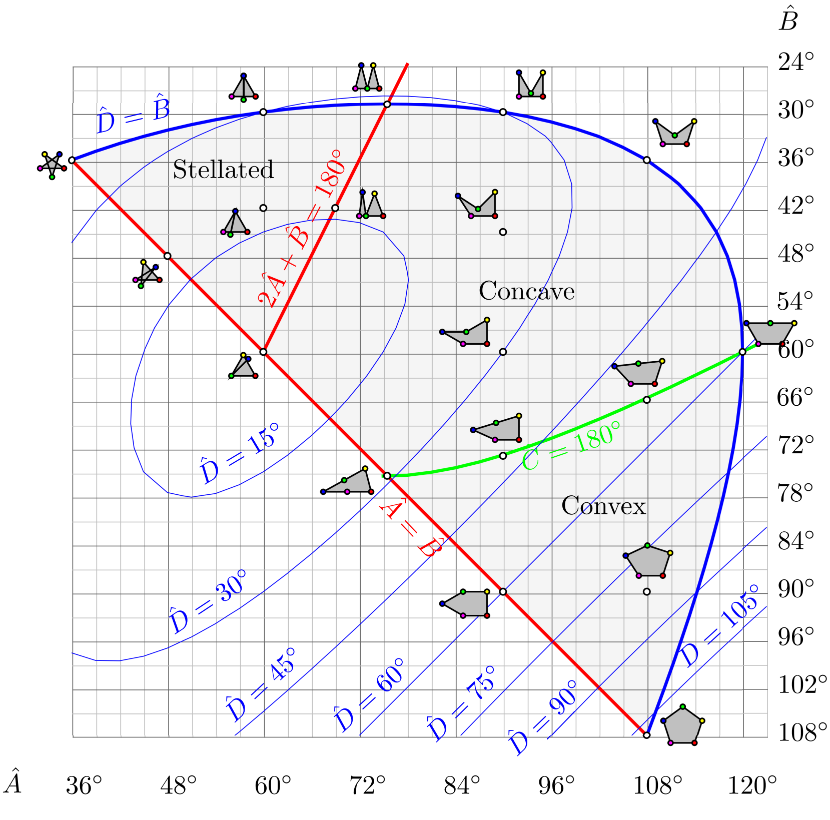

Figure 20: Triangles with equal area

4.7 Equiangular Pentagons

The area of a convex equiangular pentagon is a function of the lengths of the sides alone, since we can express the c_1 and c_2 of Gauss's Formula as

$$c_1 = \frac{\sin 108^\circ}{2} \sum_{\circlearrowleft} AB \cdot AC$$

$$c_2 = \frac{\sin^2 108^\circ}{4} \sum_{\circlearrowleft} AB \cdot BC^2 \cdot CD$$

4.8 Brocard Pentagons

Definition 14. *A **Brocard Point** of a pentagon is a point X such that all angles AXB $\circlearrowleft$ are equal. The angle is called the **Brocard angle**. A pentagon with a Brocard point is called a **Brocard pentagon**.*

Theorem 66. *If a pentagon has a Brocard point, it is unique.*

Proof. Suppose there are two Brocard points, X and X', with Brocard angles ω and ω'. If $\omega = \omega'$, then X' must be the intersection of AX and BX, and hence $X = X'$. Suppose then that $\omega \neq \omega'$. If $X \neq X'$, then in must lie in one of the triangles $\triangle ABX$ $\circlearrowleft$. If it lies, for instance, in triangle ABX, then $\omega = \angle XAB > \angle X'AB = \omega'$. But also, $\omega = \angle XBC < \angle X'BC = \omega'$. We thus have both $\omega > \omega'$ and $\omega < \omega'$, which cannot be. Similar contradictions are obtained if X lies in any of the triangles ABX $\circlearrowleft$, and hence $\omega \neq \omega'$ cannot be true. Thus $\omega = \omega'$, and hence $X = X'$. □

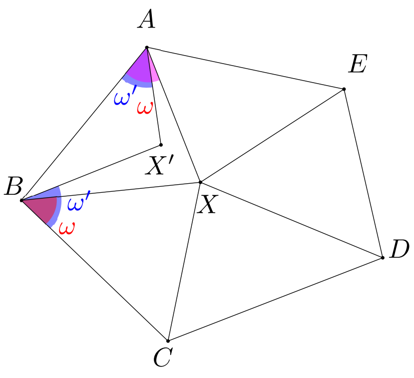

Figure 21: If a pentagon has a Brocard point, it is unique.

Theorem 67. *In a Brocard pentagon, AB $\circlearrowleft$ is tangent to $\odot XBC$.*

Theorem 68. *Let $ABCDE$ be a Brocard pentagon with Brocard point X, and $A' \neq X$ be any point on $\odot EAX$ inside $ABCDE$. Join EA', and then let $B' = AA' \cap \odot ABX$, $C' = BB' \cap \odot BCX$, $D' = CC' \cap \odot CDX$, and $E' = DD' \cap \odot DEX$. Then:*

(1) $E' = EA' \cap \odot DEX$

(2) $ABCDE \sim A'B'C'D'E'$

(3) $A'EA = B'AB$ $\circlearrowleft$

(4) *X' is also the Borcard point of $A'B'C'D'E'$*

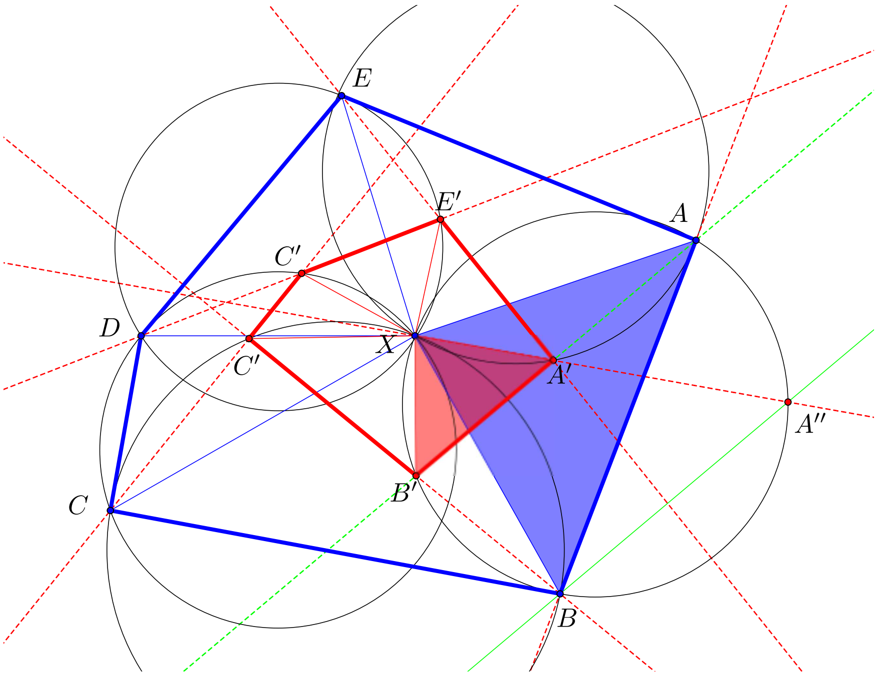

Figure 22:

Proof. Theorem 26 implies parts 1 and 2.

To prove part 3: AB is tangent to $\odot EAX$, and AA' is a chord of $\odot EAX$, so $\angle A'AB = \angle AEA'$, and (similarly $\circlearrowleft$).

I only prove the part 4 in the case where A' is in $\triangle ABX$.

Extend XA to meet $\odot ABX$ in A''. Then,

- $\angle B'AB = \angle A'EA$ (already proven, part 3 of this theorem).
- $\angle A'EA = \angle A'XA$ (angles in the same segment in circle EAX on chord $A'A$).
- $\angle AXA' = \angle AXA'' = \angle ABA''$ (angles in the same segment in circle ABX on chord AA'').

Thus $\angle B'AB = \angle ABA''$, and hence $AB' \parallel A''B$. So,

- $\angle XA'B' = \angle XA''B$ (alternate angles $AB' \parallel A''B$).
- $\angle XA''B = \angle XAB$ (angles in the same segment in circle ABX on chord XB).

Thus $\angle XA'B' = \angle XAB$.

Similarly, we can show $\angle XA'B' = \angle XAB \circlearrowleft$. But $\angle XAB \circlearrowleft$ are all equal to the Brocard angle, so $XA'B' \circlearrowleft$ must all be equal to the Brocard angle. Thus, X is also the Brocard point of $A'B'C'D'E'$. $\square$

These ideas can be extended to general polygons. See [1].

4.9 Classification By Subangles

Theorem 69 (Subangle Classification)**.** *If the subangles of a pentagon satisfies certain relationships, the pentagon is special.*

(1) *If* $A_2 = B_3 = E_1$ ↻*, the pentagon is cyclic.*

(2) *If* $A_2 = C_1 = D_3$ ↻*, the pentagon is paradiagonal.*

(3) *If* $A_1 = A_2 = A_3$ ↻*, the pentagon is regular.*

(4) *If* $A_3 = C_1$ ↻*, the pentagon is equilateral.*

Quadrilaterals	Pentagon
Parallelograms	Mediocentric
Kites	Orthocentric
Cyclic with two opposite angles right angles	Tangent

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