

Reference for Discrete Calculus and Functional Equations

Version 1.5

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Contents

1. Introduction	8
2. Functions and Sequences	9
2.1. Discrete Power Function	9
2.1.1. Definitions	11
2.1.2. Properties	12
2.1.3. Conversion to Standard Powers	13
2.2. Discrete Trigonometric Functions	14
2.2.1. Definition	14
2.2.2. Properties	14
2.3. Fibonacci Function	15
2.3.1. Definition	15
2.3.2. Properties of the Fibonacci Function	15
2.4. Derangement Function	15
2.5. Gamma Sum	16
2.6. Tangent Sum	16
2.7. Functions Involving Exponential Expansions of Hyperbolic Functions	17
2.7.1. Definitions	17
2.7.2. Properties	17
2.8. Exponential Sums	18
2.8.1. Definitions	18
2.8.2. Graphs	18
2.9. Functions Related to the Digamma Function	20
2.10. The Super Exponential Function and Super Logarithm	20
I. Discrete Calculus Tables	21
3. Arithmetic Differences and Sums	23
3.1. Definition and Properties	23
3.1.1. Definition	23
3.1.2. Properties of Operators	23
3.2. Differences	26
3.2.1. Powers	26
3.2.2. Exponential Functions	26
3.2.3. Trigonometric Functions	27
3.2.4. Inverse Trigonometric Functions	27
3.2.5. Hyperbolic Functions	28
3.2.6. Exponential Sums	28
3.2.7. Discrete Additive Trigonometric Functions	28
3.2.8. Gamma and Related Functions	29
3.2.9. More Exponential Forms	29
3.2.10. The Discrete Square Root	30
3.2.11. Forms involving trigonometric and exponential functions	31
3.2.12. Forms involving the inverse tangent function	31
3.3. Sums	32
3.3.1. Basics	32

3.3.2.	Trigonometric Functions	33
3.3.3.	Hyperbolic Functions	34
3.3.4.	Discrete Trigonometric Functions	35
3.3.5.	Exponential Sums	35
3.3.6.	Gamma Functions	36
3.3.7.	Forms involving $ax + b$	36
3.3.8.	Forms involving the sum of squares	37
3.3.9.	Forms involving general quadratic expressions	37
3.3.10.	Discrete Square Root	37
3.3.11.	Forms involving the exponential function	38
3.4.	Taylor Series	38
3.4.1.	Definition	38
3.4.2.	Table	39
3.5.	Analogs	40
3.5.1.	Definition	40
3.5.2.	Properties	41
3.5.3.	Table of Analogs	41
3.6.	Miscellaneous Difference Identities	41
3.7.	Difference Equations	41
3.7.1.	41	
3.7.2.	42	
3.7.3.	42	
4.	Geometric Differences and Sums	43
4.1.	Definitions and Properties	43
4.1.1.	Definitions	43
4.1.2.	Properties of Operators	43
4.2.	Differences	44
4.2.1.	Basics	44
4.3.	Sums	44
4.3.1.	Basics	44
4.3.2.	Trigonometric Functions	44
4.4.	Analogs	45
4.4.1.	Properties	45
4.4.2.	Table of Analogs	45
5.	Arithmetic Quotients and Products	46
5.1.	Definitions and Properties	46
5.1.1.	Definition	46
5.1.2.	Properties of Operators	46
5.1.3.	Relation to Additive Operators	46
5.2.	Quotients	47
5.2.1.	Basics	47
5.2.2.	Trigonometric Functions	47
5.2.3.	Hyperbolic Functions	47
5.2.4.	Discrete Additive Trigonometric Functions	48
5.2.5.	Gamma and Related Functions	48
5.3.	Products	48
5.3.1.	Basics	48
5.3.2.	Miscellaneous	48
5.4.	Taylor Series	49
5.5.	Miscellaneous Identities	49
5.6.	Quotient Equations	49
5.7.	Analogs	49
5.7.1.	Analogs of Elementary Functions	49

Contents

6. Geometric Quotients and Products	50
6.1. Definition	50
6.2. Quotients	50
6.2.1. Basics	50
7. Generalised Operators	51
7.1. Definitions	51
7.2. Theorems	51
7.3. Standard Functions of Difference Operators	51
7.4. Definitions	51
7.5. Theorems	51
7.6. Analogous Functions	52
7.7. Table	52
8. Trigonometric Equations	53
 II. Transforms 54	
9. The z-Transform	55
9.1. Definition and Properties	55
9.2. Transform Pairs	56
9.2.1. Basics	57
9.2.2. Trigonometric Functions	59
9.2.3. Hyperbolic Functions	59
9.2.4. Discrete Trigonometric Functions	59
9.2.5. Gamma and Related Functions	60
9.2.6. Exponential Sums	61
9.2.7. Orthogonal Polynomials	61
9.2.8. Sums of Orthogonal Polynomials	62
10. Binomial Transforms	63
10.1. Definition	63
10.2. Properties	63
10.3. Pairs	63
10.3.1. Basics	63
10.4. Pairs (Inverse)	64
10.4.1. Basics	64
10.4.2. Discrete Trigonometric Functions	64
10.4.3. Gamma and Related Functions	64

Preface

Version 1

This document started off as notes for my own work. After getting too frustrated by not being able to find auxiliary formulas in my (many) notebooks and having to derive them repeatedly, I decided to compile a neat reference for myself. This is very much a work in progress. This version contains some gaping holes and is admittedly sloppy in many respects. In particular, watch out for these issues:

- The discrete Taylor series might not converge everywhere. For example, the expansion of the Gamma function only converges to the Gamma function for integer x . In addition, the radius of convergence has been omitted everywhere.
- In many cases the constant difference has been taken as $h = 1$. This has been indicated in some places, but not in all.
- In the section on standard functions of difference operators, I have not specified all the conditions for theorems where inverse functions are involved (these work “straight out of the box” only for bijections).
- The z -transform table contains some formulas that have been formally derived, without checking whether the series (of the definition) actually converges. This has been indicated by little question marks next to the transformation arrows. The range of convergence has been omitted everywhere.

Furthermore, I used some non-standard notations, and there are many inconsistencies in style or form that make this reference hard to read.

I intend to address these in coming versions; in the mean time, even with all its flaws, some might still find this reference useful.

Version 1.1

- Added Exponential Sums to differences and sums.
- Additions to the z -transform table.
- Added Binomial Transform pairs.

Version 1.2

- Expanded the section on the discrete power functions.
- Expanded the section that explain the sue of constants in the table.
- Added forms involving the following expressions to the sum $(x + h)$ tables :
 - $ax + b$
 - $x^2 + a^2$
- Updated all the graphs.
- Reorganized slightly, and fixed some typos.
- Added a few examples, explanations, and additional notations in the sum $(x + h)$ tables.

Version 1.3

- Made several corrections.
- Added the chain and substitution rules for arithmetic differences.
- Added table of functions for reference.
- Expanded the introduction somewhat.
- Added definition for arithmetic difference analogs.
- Added rules for manipulating arithmetic difference analogs.
- Added several new entries, including several functions whose sums can be expressed as the sum of $E(x) = 1/(e^{ix} + 1)$.
- Expressed the G function (sum of the Gamma function) as a product of known functions, and replaced its notation. The notation $G(x)$ is now used for the Barnes G -function.

Version 1.4

- Made several corrections.
- Made some minor additions to many of the tables.

- Added the tangent sum function. There are still many details to sort out for this and related functions (\cot , \sec , \csc , their hyperbolic counterparts, $1/(e^x+1)$, and so on), and hence these sections are still messy. These will be cleaned up as the details become clear.
- Replaced some of the statements on periodic, odd, and even functions with precise versions. The previous ones were only correct up to a periodic function.
- Added the derangement function (expressed in terms of the incomplete gamma function), as well as some related Taylor series.
- Since I included the definitions of analog functions, I discovered that the intuitive notion of analogs did not correspond to the definition. Thus, the analogs of $\ln x$ and $\text{atan } x$ have been removed / replaced. These might re-appear if the definition of analog functions is suitably adjusted.
- Made many statements on the z -transform more precise.
- Made some notations more consistent with standard notation.

Version 1.5

- Made a minor correction to the binomial law for discreet powers.

Contributions

The indefinite sum of $\tan x$ was contributed by “Anixx” from a question on *MathOverflow*, with further details supplied by Gerald Edgar. From this the indefinite sums of several other functions that depend on E were derived. The question is here:

<http://www.mathoverflow.net/questions/41011/what-is-the-indefinite-sum-of-tanx>.

1. Introduction

This reference gives tables and formulas that should be useful for dealing with functional equations, particularly difference equations. It is divided in two main parts.

The first part deals with various discrete calculi, with main operators summarised in the table below:

Calculus	Main operator	Inverse
Arithmetic difference calculus	$\Delta f(x) = f(x + h) - f(x)$	\sum
Geometric difference calculus	$P^{-1} f(x) = f(hx) - f(x)$	$P f(x)$
Arithmetic quotient calculus	$Q f(x) = f(x + h)/f(x)$	\prod
Geometric quotient calculus	$R f(x) = f(hx)/f(x)$	R^{-1}
Generalised difference calculus	$\Delta_g f(x) = f(g(x)) - f(x)$	Δ_g^{-1}
Generalised quotient calculus	$P_g f(x) = f(g(x))/f(x)$	P_g^{-1}

For each¹ we give the following:

- definitions and basic properties,
- differences or quotients,
- sums or products,
- Taylor series, and
- functions analog to elementary functions in standard calculus.

The second part of this reference deals with transforms. Properties and tables of the following transforms are given:

- the z -transform, and
- the binomial transform.

Definitions and properties of most functions used in this reference can be found in any good mathematical handbook. The next section summarises functions (definitions and properties) for those functions that do not appear in [NHMF], and give references for those that do.

¹At this stage, some sections are still incomplete.

2. Functions and Sequences

In the table below, references without labels refer to sections in this text.

Notation	Name	Reference
$a^{(x)}$	Discrete power function	2.1
$B_n(x)$	Bernoulli polynomial	[NHMF] 24.2.1
$\text{Bs } x$	(Binary) exponential sum	2.8.1
$\text{Cds } x$	Discrete cosine sum	2.8.1
$\text{cosd } x$	Discrete cosine	2.2
$\text{Cs } x$	Cosine sum	2.8.1
$d(x)$	Derangement function	2.4
$E(x)$		2.7
$\epsilon(x)$	Sum of $E(2x)$	2.7
$\epsilon_i(x)$	Sum of $E(2ix)$	2.7
$E_n(x)$	Euler polynomial	[NHMF] 24.2.6
G	Barnes' G -function or double Gamma function	[NHMF] 5.17.1
$\Gamma(x)$	Gamma function	[NHMF] 5.2.1
$\Gamma^{(-1)}(x)$	Gamma sum	2.5
$\gamma(x, a)$	Lower incomplete Gamma function	[NHMF] 8.2.1
$\Gamma(x, a)$	Upper incomplete Gamma function	[NHMF] 8.2.2
$\gamma^*(x, a)$		[NHMF] 8.2.7
$H_k(x)$	Hermite polynomial	[NHMF] 18.3
$I_t(x, a)$	Normalised incomplete Beta function	[NHMF] 8.17.2
$J_k(x)$	Bessel polynomial	[NHMF] 18.3
$K(x)$		
$L_k(x)$	Laguerre polynomial	[NHMF] 18.3
$P(x, a)$	Normalised lower incomplete Gamma function	[NHMF] 8.2.4
$P_k(x)$	Legendre polynomial	[NHMF] 18.3
ψ	Digamma or Psi function	[NHMF] 5.2.2
$\psi^{(n)}$	Polygamma function, the n th derivative of ψ	[NHMF] 5.15
$\psi_{e^{2i}}(x)$	q -digamma function, with $q = e^{2i}$	
$s(n, k)$	Stirling numbers of the first kind	[NHMF] 26.8(i)
$S(n, k)$	Sterling numbers of the second kind	[NHMF] 26.8(i)
$\text{sexp}_a x$	Super exponential function	2.10
$\text{sind } x$	Discrete sine	2.2
$\text{Sds}(x)$	Discrete sine sum	2.8.1
$\text{slog}_a x$	Super logarithm	2.10
$\text{Ss}(x)$	Sine sum	2.8.1
$T_k(x)$	Chebyshev polynomial	[NHMF] 18.3
$W(x)$	Lambert W function	[NHMF] 4.13.1

2.1. Discrete Power Function

The discrete power function has properties in discrete calculus analogous to the standard power function in standard calculus. For example, see (3.2.7). The function is also sometimes called the *falling factorial*.

2. Functions and Sequences

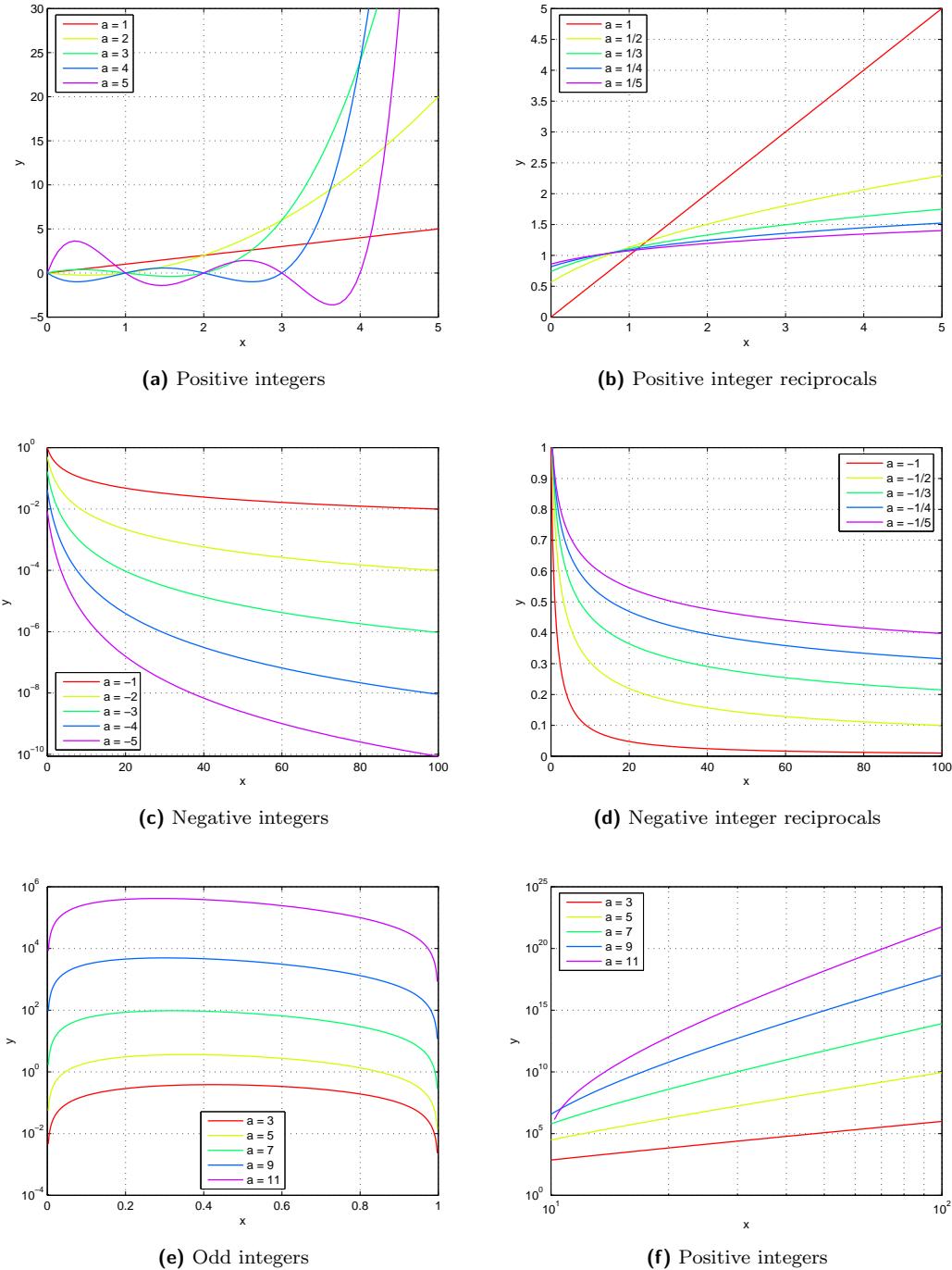


Figure 2.1.: Discrete Power Function. $y = x^{(a)}$

2.1.1. Definitions

$$x^{\langle a \rangle} = \frac{\Gamma(x+1)}{\Gamma(x+1-a)} \quad x \notin \mathbb{Z}^- \quad x-a \notin \mathbb{Z}^- \quad (2.1.1)$$

$$x^{\langle a \rangle_h} = h^a \frac{\Gamma\left(\frac{x}{h} + 1\right)}{\Gamma\left(\frac{x}{h} + 1 - a\right)} \quad \frac{x}{h} \notin \mathbb{Z}^- \quad \frac{x}{h} - a \notin \mathbb{Z}^- \quad (2.1.2)$$

In cases where the Gamma function is infinite, the limit is taken if it exists:

$$x^{\langle a \rangle} = \lim_{t \rightarrow x} \frac{\Gamma(t+1)}{\Gamma(t+1-a)} \quad (2.1.3)$$

$$x^{\langle a \rangle_h} = \lim_{t \rightarrow x} h^a \frac{\Gamma\left(\frac{t}{h} + 1\right)}{\Gamma\left(\frac{t}{h} + 1 - a\right)} \quad (2.1.4)$$

Note that $x^{\langle a \rangle}$ is just the special case of $x^{\langle a \rangle_h}$ when $h = 1$.

Alternative notation The following notation is commonly used instead of the $x^{\langle n \rangle}$ presented above: x^a , $x^{\langle a \rangle}$, $(x)_a$.

Examples

$$\begin{aligned} 6^{\langle 3 \rangle} &= \frac{\Gamma(6+1)}{\Gamma(6+1-3)} = \frac{720}{6} = 120 & 6^{\langle 3 \rangle_2} &= 2^3 \frac{\Gamma(\frac{6}{2}+1)}{\Gamma(\frac{6}{2}+1-3)} = 8 \cdot \frac{6}{1} = 48 \\ \left(-\frac{1}{2}\right)^{\langle -1 \rangle} &= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} = \frac{\sqrt{\pi}}{\frac{3}{2}\sqrt{\pi}} = \frac{2}{3} & \left(-\frac{1}{2}\right)^{\langle -1 \rangle_{\frac{1}{2}}} &= 2^{-1} \frac{\Gamma(2)}{\Gamma(5)} = \frac{1}{2} \cdot \frac{1}{24} = \frac{1}{48} \end{aligned}$$

2.1.2. Properties

For integer n :

$$x^{\langle n \rangle} = x(x-1)(x-2)\dots(x-n+1) \quad n > 0 \quad (2.1.5)$$

$$x^{\langle -n \rangle} = \frac{1}{(x+1)(x+2)\dots(x+n)} \quad n > 0 \quad (2.1.6)$$

$$x^{\langle 0 \rangle} = 1 \quad (2.1.7)$$

$$x^{\langle n \rangle_h} = x(x-h)(x-2h)\dots(x-(n-1)h) \quad n > 0 \quad (2.1.8)$$

$$x^{\langle -n \rangle_h} = \frac{1}{(x+h)(x+2h)\dots(x+nh)} \quad n > 0 \quad (2.1.9)$$

$$x^{\langle 0 \rangle_h} = 1 \quad (2.1.10)$$

$$\prod_{k=0}^{n-1} \left(x - \frac{k}{n} \right)^{\langle \frac{1}{n} \rangle} = x \quad (2.1.11)$$

$$x^{\langle n \rangle} = (-1)^{n+1} \Gamma(x+1) \Gamma(n-x) \sin \pi x \quad x \notin \mathbb{Z} \quad (2.1.12)$$

$$(-x)^{\langle n \rangle} = (-1)^n (x+n-1)^{\langle n \rangle} \quad (2.1.13)$$

$$(-x)^{\langle n+\frac{1}{2} \rangle} = (-1)^n \left(x + n - \frac{1}{2} \right)^{\langle n+\frac{1}{2} \rangle} \tan \pi x = 0 \quad (2.1.14)$$

$$1^{\langle n \rangle} = \begin{cases} 0 & n > 1 \\ 1 & n = 1 \\ \frac{1}{(n+1)!} & n < 1 \end{cases} \quad (2.1.15)$$

$$0^{\langle n \rangle} = \begin{cases} 0 & n > 0 \\ 1 & n = 0 \\ \frac{1}{n!} & n < 0 \end{cases} \quad (2.1.16)$$

$$(x+y)^{\langle n \rangle} = \sum_{k=0}^n \binom{n}{k} x^{\langle k \rangle} y^{\langle n-k \rangle} \quad (\text{Binomial Theorem}) \quad (2.1.17)$$

$$n^{\langle n \rangle} = n! \quad (2.1.18)$$

Note that (2.1.7) and (2.1.10) also holds when $x = 0$, that is, $0^{\langle 0 \rangle} = 1$.

For real a :

$$x^{\langle a \rangle} (x-a)^{\langle b \rangle} = (x-b)^{\langle a \rangle} x^{\langle b \rangle} = x^{\langle a+b \rangle} \quad (2.1.19)$$

$$x^{\langle -a \rangle} = \frac{1}{(x+a)^{\langle a \rangle}} \quad (2.1.20)$$

$$x^{\langle a \rangle_h} (x-ah)^{\langle b \rangle_h} = (x-bh)^{\langle a \rangle_h} x^{\langle b \rangle_h} = x^{\langle a+b \rangle} \quad (2.1.21)$$

$$x^{\langle -a \rangle_h} = \frac{1}{(x+ah)^{\langle a \rangle_h}} \quad (2.1.22)$$

$$x^{\langle a \rangle} = -\Gamma(x+1) \Gamma(a-x) \sin(a-x)\pi \quad a \notin \mathbb{Z} \quad (2.1.23)$$

$$(-x)^{\langle a \rangle} = (x+a-1)^{\langle a \rangle} (\cos a\pi + \sin a\pi \tan \pi x) \quad a \notin \mathbb{Z} \quad (2.1.24)$$

$$1^{\langle a \rangle} = \Gamma(a-1) \sin \pi a \quad a \notin \mathbb{Z}^- \quad (2.1.25)$$

$$0^{\langle a \rangle} = -\Gamma(a) \sin \pi a \quad a-1 \notin \mathbb{Z}^- \quad (2.1.26)$$

$$x^{\langle x \rangle} = \Gamma(x+1) \quad (2.1.27)$$

The following approximation mimics the formula $(x^a)^{1/a} = x$ that we have for standard powers:

$$(x^{\langle a \rangle})^{\langle 1/a \rangle} \approx m_a x + c_a \text{ for } x > x_a \quad (2.1.28)$$

2. Functions and Sequences

The following results have been obtained numerically for $1 \leq a \leq 2.5$ and $x_a \leq x \leq 5$. It is possible that it holds for larger x and larger a .

$$x_a \approx 1.0519a - 1.5814 \quad (2.1.29)$$

$$m_a \approx 0.062898a^2 - 0.22811a + 1.1753 \quad (2.1.30)$$

$$c_a \approx -0.2793a^2 + 0.44094a - 0.152 \quad (2.1.31)$$

The curves in Figure 2.2 gives values for x_a , m_a and c_a when $0.3 \leq a \leq 1$.

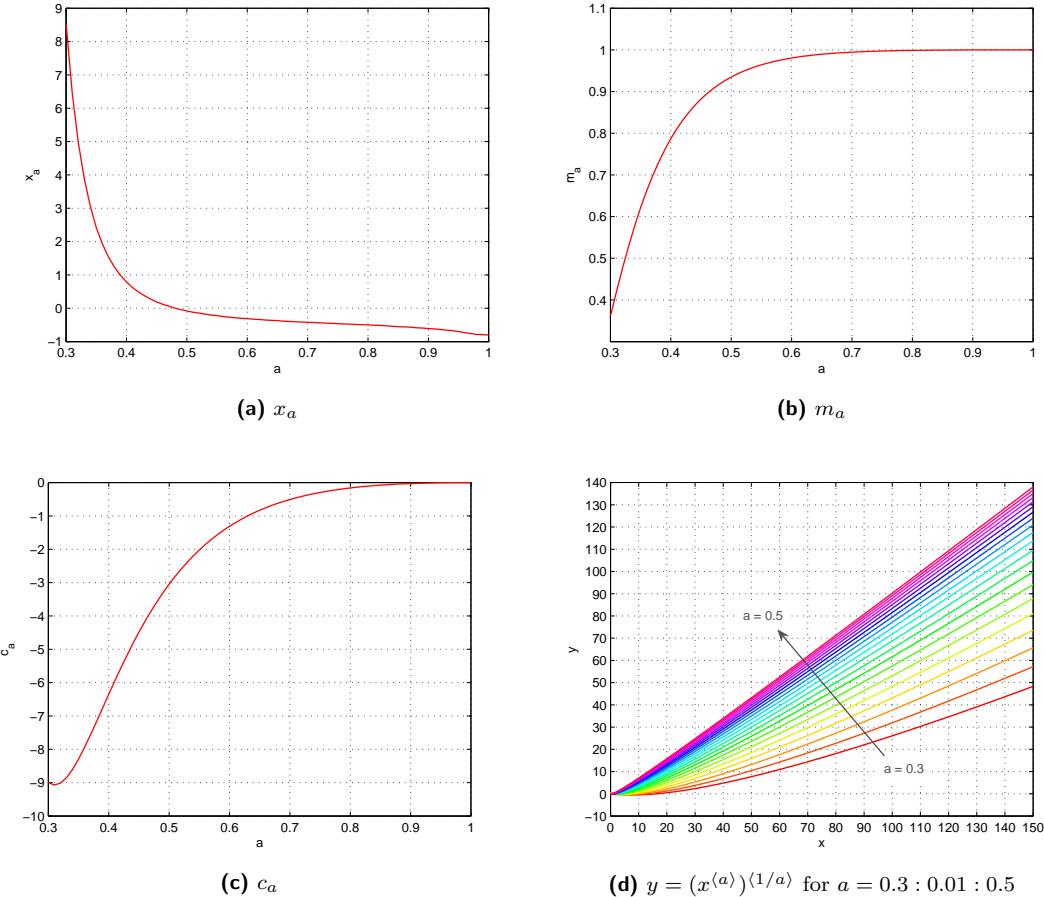


Figure 2.2.: Curves used in the inverse power approximation.

2.1.3. Conversion to Standard Powers

For integer n ,

$$x^{\langle n \rangle} = \sum_{k=1}^n s(n, k)x^k \quad (2.1.32)$$

$$x^n = \sum_{k=1}^n S(n, k)x^{\langle k \rangle} \quad (2.1.33)$$

2.2. Discrete Trigonometric Functions

Discrete trigonometric functions are the discrete counterparts of standard trigonometric functions. For example, see (3.2.40) and (3.2.39).

2.2.1. Definition

$$\text{sind } x = 2^{x/2} \sin\left(\frac{\pi x}{4}\right) \quad (2.2.1)$$

$$\text{cosd } x = 2^{x/2} \cos\left(\frac{\pi x}{4}\right) \quad (2.2.2)$$

$$\text{cisd } x = 2^{x/2} \text{cis}\left(\frac{\pi x}{4}\right) \quad (2.2.3)$$

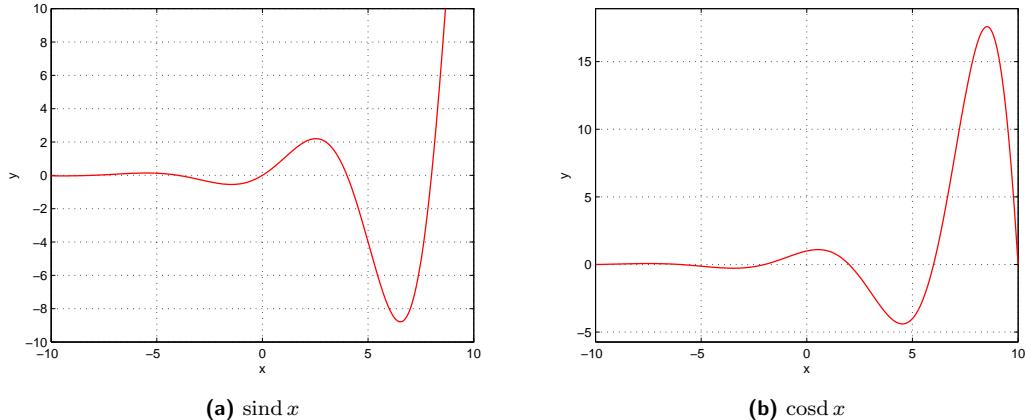


Figure 2.3.: Discrete Trigonometric Functions

2.2.2. Properties

$$\frac{\text{sind } x}{\text{cosd } x} = \tan \frac{\pi x}{4} \quad (2.2.4)$$

$$\frac{\text{cosd } x}{\text{sind } x} = \cot \frac{\pi x}{4} \quad (2.2.5)$$

$$\text{cisd } x = (2i)^{\frac{x}{2}} \quad (2.2.6)$$

$$\text{sind}(x + 8n) = 16^n \text{sind } x \quad (2.2.7)$$

$$\text{cosd}(x + 8n) = 16^n \text{cosd } x \quad (2.2.8)$$

$$\text{sind}(x + y) = \text{sind}(x) \text{cosd}(y) + \text{cosd}(x) \text{sind}(y) \quad (2.2.9)$$

$$\text{sind}(x - y) = 2^y \text{sind}(x) \text{cosd}(y) - 2^y \text{cosd}(x) \text{sind}(y) \quad (2.2.10)$$

$$\text{cosd}(x + y) = \text{cosd}(x) \text{cosd}(y) - \text{sind}(x) \text{sind}(y) \quad (2.2.11)$$

$$\text{cosd}(x - y) = 2^y \text{cosd}(x) \text{cosd}(y) + 2^y \text{sind}(x) \text{sind}(y) \quad (2.2.12)$$

$$\text{sind}^2(x) + \text{cosd}^2(x) = 2^x \quad (2.2.13)$$

$$\text{sind}(2x) = 2 \text{sind}(x) \text{cosd}(x) \quad (2.2.14)$$

$$\text{cosd}(2x) = \text{cosd}^2(x) - \text{sind}^2(x) \quad (2.2.15)$$

$$= 2 \text{cosd}^2(x) - 2^x \quad (2.2.16)$$

$$= 2^x - 2 \text{sind}^2(x) \quad (2.2.17)$$

2.3. Fibonacci Function

2.3.1. Definition

$$\mathcal{F}(x) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^x - \left(\frac{1-\sqrt{5}}{2} \right)^x \right] \quad (2.3.1)$$

2.3.2. Properties of the Fibonacci Function

$$\mathcal{F}(x) = \mathcal{F}(x-1) + \mathcal{F}(x-2) \quad (2.3.2)$$

$$\mathcal{F}(x) = \sum_{k=0}^n \binom{n}{k} \mathcal{F}(x-k-m) \quad (2.3.3)$$

2.4. Derangement Function

The derangement function is defined as:

$$d(x) = \frac{\Gamma(x+1, -1)}{e}. \quad (2.4.1)$$

For integer n ,

$$d(n) = n! \sum_{k=0}^n \frac{(-1)^k}{k!} \quad (2.4.2)$$

The notation $!n$ sometimes used to denote this function. For integer $n > 1$, the following is useful for approximating $d(n)$:

$$d(n) = \left\lfloor \frac{n! - 2}{e} \right\rfloor + 1 \quad (2.4.3)$$

k	0	1	2	3	4	5	6	7
$d(k)$	1	0	1	2	9	44	265	1854

$$d(x+1) - d(x) = xd(x) + (-1)^{x+1} \quad (2.4.4)$$

$$d(x+1) + d(x) = (x+2)d(x) + (-1)^{x+1} \quad (2.4.5)$$

2.5. Gamma Sum

The Gamma Sum is given by:

$$\Gamma^{\langle -1 \rangle}(x) = e^{-1}(-1)^{1-x}\Gamma(x)\Gamma(1-x,-1) = (-1)^{1-x}\Gamma(x)d(-x) \quad (2.5.1)$$

It satisfies the following recurrence relation:

$$\Gamma^{\langle -1 \rangle}(x+1) = \Gamma^{\langle -1 \rangle}(x) + \Gamma(x) \quad (2.5.2)$$

2.6. Tangent Sum

The tangent sum function is defined by

$$T_a(x) = -\frac{1}{a} \sum_{k=1}^{\infty} \psi\left(\frac{\pi}{a}\left(k-\frac{1}{2}\right)-x+\frac{1}{a}\right) + \psi\left(\frac{\pi}{a}\left(k-\frac{1}{2}\right)+x\right) - \psi\left(\frac{\pi}{a}\left(k-\frac{1}{2}\right)+\frac{1}{a}\right) - \psi\left(\frac{\pi}{a}\left(k-\frac{1}{2}\right)\right) \quad (2.6.1)$$

The shorthand $T(x)$ is used for $T_1(x)$.

$$\Delta_{x,1} T_a(x) = T_a(x+1) - T_a(x) = \tan ax \quad (2.6.2)$$

$$\Delta_{x,1} T_a\left(x + \frac{\pi k}{a}\right) = T_a\left(x + 1 + \frac{\pi k}{a}\right) - T_a\left(x + \frac{\pi k}{a}\right) = \tan ax \quad k \in \mathbb{Z} \quad (2.6.3)$$

$$\Delta_{x,\pi} T(x) = T(x+\pi) - T(x) = -\pi \cot \pi\left(x + \frac{\pi}{2}\right) \quad (2.6.4)$$

$$T\left(x - \frac{1}{2}\right) = T\left(\frac{1}{2} - x\right) \quad (2.6.5)$$

$$\Delta_{x,1}^2 \Delta_{x,\pi} T\left(\frac{x}{2}\right) = \csc x \quad (2.6.6)$$

$$T(1) = T(0) = 0 \quad (2.6.7)$$

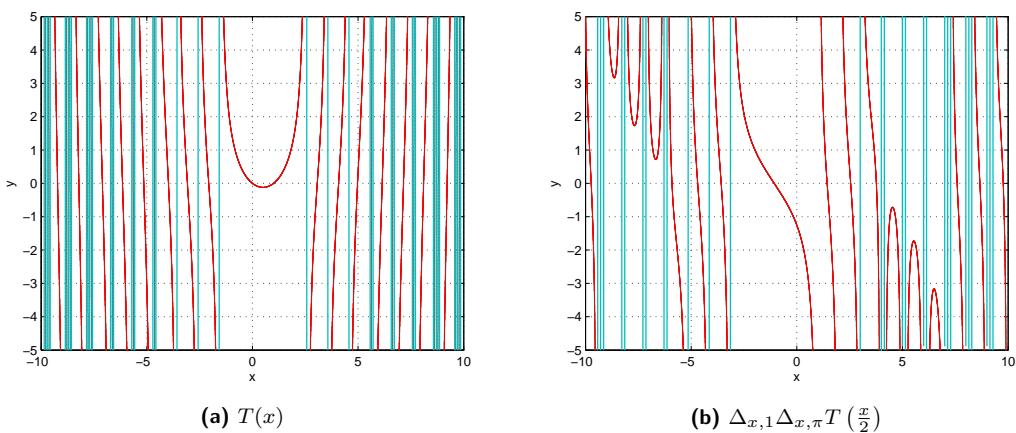


Figure 2.4.: Tangent sum and Cosecant sum. Blue shows asymptotes. As can be seen, the density of poles increase rapidly as $|x|$ increases.

2.7. Functions Involving Exponential Expansions of Hyperbolic Functions

2.7.1. Definitions

The E function is defined as:

$$E(x) = \frac{1}{e^{ix} + 1} \quad (2.7.1)$$

The sum of $E(2x)$ is given by $\epsilon(x)$, where

$$\epsilon(x) = -\frac{1}{2} (iT(x) + x) \quad (2.7.2)$$

The following is a place-holder definition. For the moment, it allows us to express some functions in terms of a single function. Once a suitable, proper definition is found, it will replace this.

$$\epsilon_i(x) = \sum_{t=0}^{x-1} E(2it) \quad (2.7.3)$$

2.7.2. Properties

$$E(x + 2\pi) = E(x) \quad (2.7.4)$$

$$E(-x) = e^{ix} E(x) \quad (2.7.5)$$

$$E(2x) = \frac{E(x + \pi/2) + E(x - \pi/2)}{2} \quad (2.7.6)$$

$$\frac{1}{e^{ix} + e^{iy}} = e^{-iy} E(x - y) = e^{-ix} E(y - x) \quad (2.7.7)$$

Trigonometric Functions Expressed in terms of E

$$\sin x = \frac{i}{E(x + \pi) - E(x)} \quad (2.7.8)$$

$$\cos x = \frac{i}{E(x + \pi/2) - E(x - \pi/2)} \quad (2.7.9)$$

$$\tan x = E(2x)i + i \quad (2.7.10)$$

$$\cot x = i - 2iE(2x + \pi) \quad (2.7.11)$$

$$\sec x = iE(x + \pi/2) - iE(x - \pi/2) \quad (2.7.12)$$

$$\csc x = iE(x) - iE(x + \pi) \quad (2.7.13)$$

Hyperbolic Functions Expressed in terms of E

$$\sinh x = \frac{1}{E(ix + \pi) - E(ix)} \quad (2.7.14)$$

$$\cosh x = \frac{i}{E(ix - \pi/2) - E(ix + \pi/2)} \quad (2.7.15)$$

$$\tanh x = 2E(2ix) + 1 \quad (2.7.16)$$

$$\coth x = 2E(2ix + \pi) - 1 \quad (2.7.17)$$

$$\operatorname{sech} x = iE(ix + \pi/2) - iE(ix - \pi/2) \quad (2.7.18)$$

$$\operatorname{csch} x = E(ix + \pi) - E(ix) \quad (2.7.19)$$

2.8. Exponential Sums

2.8.1. Definitions

$$Bs(x) = \sum_{k=0}^{\infty} \frac{2^{x-k}}{x-k} \quad (2.8.1)$$

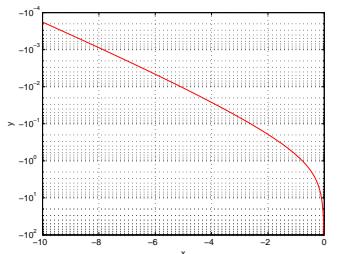
$$Ss(x) = \sum_{k=0}^{\infty} \frac{\sin(x-k)}{x-k} \quad (2.8.2)$$

$$Cs(x) = \sum_{k=0}^{\infty} \frac{\cos(x-k)}{x-k} \quad (2.8.3)$$

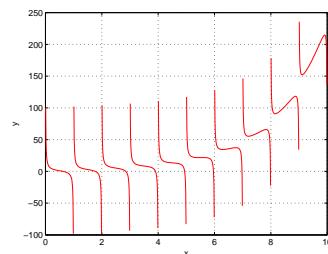
$$Sds(x) = \sum_{k=0}^{\infty} \frac{\text{sind}(x-k)}{x-k} \quad (2.8.4)$$

$$Cds(x) = \sum_{k=0}^{\infty} \frac{\text{cosd}(x-k)}{x-k} \quad (2.8.5)$$

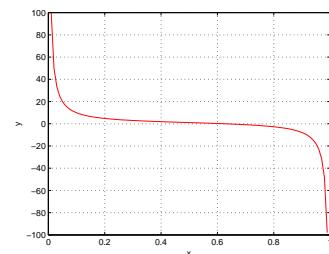
2.8.2. Graphs



(a) Negative

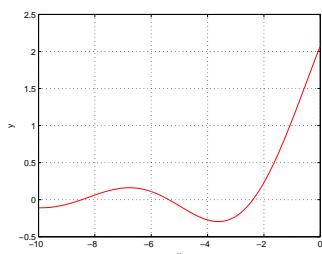


(b) Positive

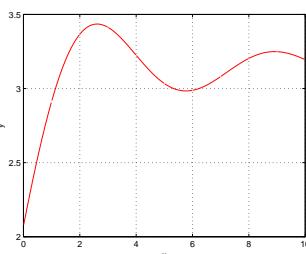


(c) Unit interval

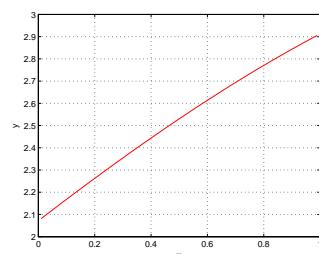
Figure 2.5.: $Bs(x)$



(a) Negative



(b) Positive



(c) Unit interval

Figure 2.6.: $Ss(x)$

2. Functions and Sequences

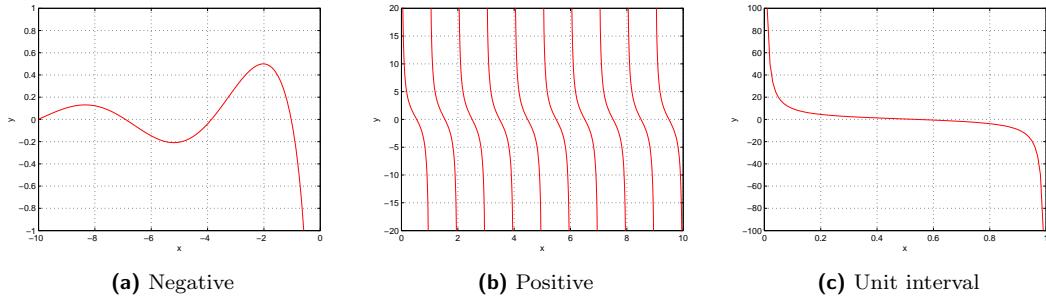


Figure 2.7.: $\text{Cs}(x)$

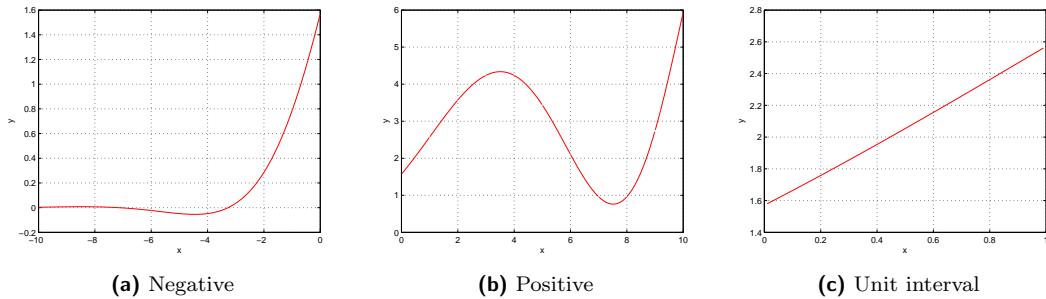


Figure 2.8.: $\text{Sds}(x)$

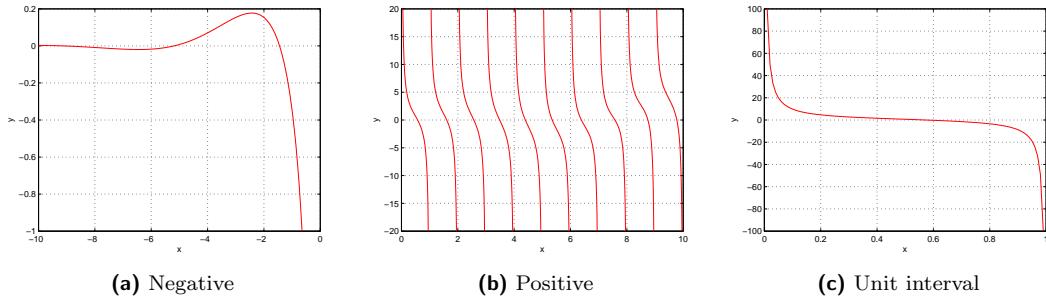


Figure 2.9.: $\text{Cds}(x)$

2.9. Functions Related to the Digamma Function

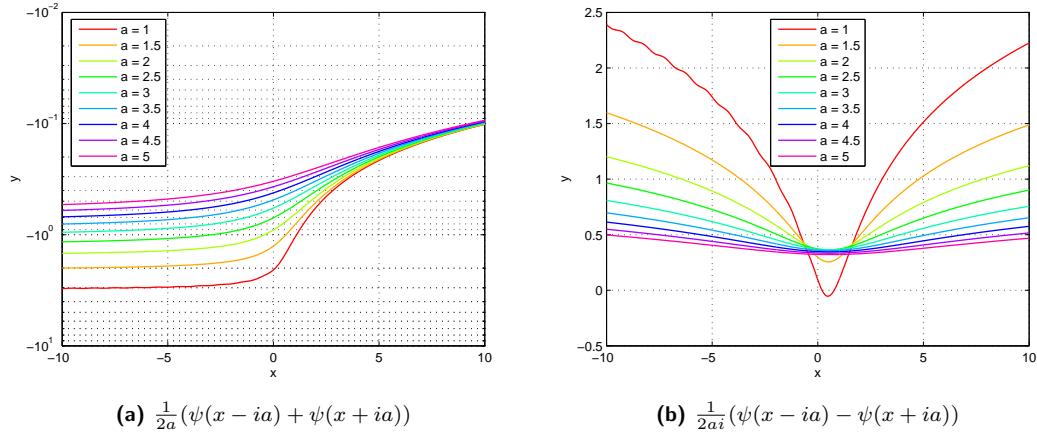


Figure 2.10.: Functions Related to the Digamma Function

2.10. The Super Exponential Function and Super Logarithm

The super exponential function is the function that satisfies

$$\text{sexp}_a(x+1) = a^{\text{sexp}_a(x)} \quad (2.10.1)$$

$$\text{sexp}_a 1 = a \quad (2.10.2)$$

For positive integer n , we have

$$\text{sexp}_a n = a^{a^{a^{\dots}}}, \quad (2.10.3)$$

with n occurrences of a .

An inverse function the super logarithm. It satisfies

$$\text{slog}_a(a^x) = \text{slog}_a(x) + 1. \quad (2.10.4)$$

Part I.

Discrete Calculus Tables

Constants

Constants can generally be replaced by any function with appropriate properties. For instance, the table gives

$$\sum a^x = \frac{a^x}{a - 1} + C.$$

Since, for any period function $\Theta(x)$ with period 1 we have $\Delta\Theta(x) = \Theta(x+1) - \Theta(x) = \Theta(x) - \Theta(x) = 0$, we can actually replace the constant C with any function with period 1, that is

$$\sum a^x = \frac{a^x}{a - 1} + \Theta(x).$$

We can also replace the a with an arbitrary periodic function (with period 1), so that we have:

$$\sum (\Theta_1(x))^x = \frac{(\Theta_1(x))^x}{\Theta_1(x) - 1} + \Theta_2(x).$$

Similarly, the table gives:

$$P^{-1} x = \frac{x}{h - 1} + C.$$

Here we can replace C with any function that satisfies $C(hx) = C(x)$. If $\Theta(x)$ is a periodic function with period 1, then $\Theta(\log_h x)$, satisfies this condition, so:

$$P^{-1} x = \frac{x}{h - 1} + \Theta(\log_h x).$$

In the general case, the constant for operator Δ_g can be replaced by $\Theta(I_\Delta g(x))$, where Θ is any periodic function with period 1 (see formula 7.5.1).

3. Arithmetic Differences and Sums

3.1. Definition and Properties

3.1.1. Definition

$$\Delta f(x) = f(x + h) - f(x) \quad (3.1.1)$$

$$\Sigma = \Delta^{-1} \quad (3.1.2)$$

Notation When x or x and h are not clear from the context, they can be specified with the symbols: $\Delta_{x,h}f(x)$ and $\Sigma_{x,h}f(x)$. The notation can also mimic that of standard calculus:

$$\frac{\Delta}{\Delta x} f(x) \quad \sum f(x) \Delta x \quad (3.1.3)$$

The n th difference of a function f can be denoted $f^{\langle n \rangle}(x) = \Delta^n f(x)$, and is the function obtained when applying the difference operator n times. The notation $\Delta f(x)|_{x=a}$ is used to denote $\Delta f(x)$ evaluated at $x = a$.

The following notation is used for definite sums:

$$\sum_a^{b-h} f(x) = \sum_{k=0}^{(b-a)/k} f(a + kh) = f(a) + f(a + h) + f(a + 2h) + \dots + f(b - h)$$

where $(b - a)/h$ is an integer.

Examples

$$\begin{aligned} \Delta x^2 &= (x + 1)^2 - x^2 = 2x + 1 \\ \Delta \Gamma(x) &= \Gamma(x + 1) - \Gamma(x) = x\Gamma(x) - \Gamma(x) = (x - 1)\Gamma(x) \end{aligned} \quad \begin{aligned} \Sigma(2x + 1) &= x^2 + C \\ \Sigma(x - 1)\Gamma(x) &= \Gamma(x) + C \end{aligned}$$

3.1.2. Properties of Operators

Basic Properties Differencing

$$\Delta(af(x) + bg(x)) = a\Delta f(x) + b\Delta g(x) \quad \text{Linearity} \quad (3.1.4)$$

$$\Delta(f(x)g(x)) = f(x + h)\Delta g(x) + g(x)\Delta f(x) \quad \text{Product Rule} \quad (3.1.5)$$

$$\Delta \frac{f(x)}{g(x)} = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x + 1)} \quad \text{Quotient Rule} \quad (3.1.6)$$

$$f^{\langle n \rangle}(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(x + hk) \quad \text{nth Difference} \quad (3.1.7)$$

$$\Delta^n(f(x)g(x)) = \sum_{k=0}^n \Delta^k f(x) \Delta^{n-k} g(x + hk) \quad \text{Leibniz's Rule} \quad (3.1.8)$$

Basic Properties Summation

$$\sum_{x=a}^{b-h} af(x) + \sum_{x=a}^b bg(x) = a \sum_{x=a}^b f(x) + b \sum_{x=a}^b g(x) \quad \text{Linearity} \quad (3.1.9)$$

$$\sum_{x=a}^{b-h} f(x) = \sum_{x=a}^b f(x) \Big|_a^b \quad \text{Fundamental Theorem} \quad (3.1.10)$$

$$\sum_{x=a}^{b-h} f(x)g(x) = f(x) \sum_{x=a}^{b-h} g(x) - \sum_{x=a}^{b-h} (\Delta f(x)) \left(\sum_{y=a}^x g(y) \right) \quad \text{Summation by Parts} \quad (3.1.11)$$

$$\sum_{x=a}^{b-h} f(x)g(x) = f(b) \sum_{x=a}^{b-h} g(x) - \sum_{x=a}^{b-h} (\Delta f(x)) \left(\sum_{y=a}^x g(y) \right) \quad \text{Abel's Transformation} \quad (3.1.12)$$

$$\sum_{x=a}^{c-h} f(x) + \sum_{x=c}^{b-h} f(x) = \sum_{x=a}^{b-h} f(x) \quad (3.1.13)$$

$$\sum_{x=a}^{b-h} f(x) = - \sum_{x=b}^{a-h} f(x) \quad (3.1.14)$$

$$\sum_{x=a}^{a-h} f(x) = 0 \quad (3.1.15)$$

$$\sum_x (-1)^x f(x) = (-1)^x \sum_u f(2u) \Big|_{(x+1)/2}^{x/2} \quad (3.1.16)$$

Multiple Summations

$$\sum_{x=a}^b \sum_{y=c}^d f(x, y) = \sum_{y=c}^d \sum_{x=a}^b f(x, y) \quad (3.1.17)$$

$$\sum_{x=a}^b \sum_{y=x}^b f(x, y) = \sum_{y=a}^b \sum_{x=a}^y f(x, y) \quad (3.1.18)$$

$$\sum_{x=a}^b \sum_{y=\phi(x)}^{\phi(b)} f(x, y) = \sum_{y=\phi(a)}^{\phi(b)} \sum_{x=a}^{\phi^{-1}(y)} f(x, y) \quad \phi(x) \text{ is invertible for } x \in [a, b] \quad (3.1.19)$$

Chain Rules

$$\Delta f(x+a) = \Delta f(x) \quad (3.1.20)$$

$$\Delta_x f(-x) = -(\Delta_u f(u))_{u=-x-1} \quad (3.1.21)$$

$$\Delta_x f(mx) = \sum_{k=0}^{m-1} \Delta_u f(u + hk) \Big|_{u=mx} \quad m \in \mathbb{N} \quad (3.1.22)$$

$$\Delta_x f(g(x)) = \sum_k \Delta_u f(u + hk) \Big|_{u=g(x)} \Big|_0^{\Delta_x g(x)} \quad (3.1.23)$$

$$\Delta_x f(g(x)) = \left(\int_0^1 f'(g(x) + ht\Delta_x g(x)) dt \right) \Delta_x g(x) \quad (3.1.24)$$

Substitution Rules

$$\sum f(x+a) = \left(\sum_u f(u) \right)_{u=x+a} \quad (3.1.25)$$

$$\sum_x f(-x) = - \left(\sum_u f(u) \right)_{u=-x+1} \quad (3.1.26)$$

$$\sum_u f\left(\frac{u}{m}\right) = \sum_{k=0}^{m-1} \left(\sum_x f(x) \right)_{x=(u+hk)/m} \quad m \in \mathbb{N} \quad (3.1.27)$$

$$\sum_u f(g(u)) = \left(\sum_k \left(\sum_x f(x) \right)_{x=g(u+hk)} \Big|_0^{\Delta_x g^{-1}(x)} \right)_{x=g(u)} \quad (3.1.28)$$

Infinite Sums The following are valid if the infinite sums converge:

$$\sum f(x) = \sum_{k=1}^{\infty} f(x-k) \quad (3.1.29)$$

$$\sum f(x+a-1) = \left(\sum f(u) \right)_{u=x+a-1} = \sum_{k=a}^{\infty} f(x-k) \quad (3.1.30)$$

$$\sum f(x) = \sum_{k=0}^{\infty} f(x+k) \quad (3.1.31)$$

The following are valid if the infinite sums converge, and f is analytical for all x :

$$\frac{1}{n!} \frac{d^n}{dx^n} f(x) = \sum_{k=n}^{\infty} \frac{s(k, n)}{k!} \Delta^k f(x) \quad (3.1.32)$$

$$\frac{1}{n!} \Delta^n f(x) = \sum_{k=n}^{\infty} \frac{S(k, n)}{k!} \frac{d^k}{dx^k} f(x) \quad (3.1.33)$$

Periodic Functions Here $\Theta(x)$ is a periodic function with period h , that is, $\Theta(x+h) = \Theta(x)$.

$$\Delta\Theta(x) = 0 \quad (3.1.34)$$

$$\Delta\Theta(x)f(x) = \Theta(x)\Delta f(x) \quad (3.1.35)$$

If Θ_τ is a periodic function with any period τ (possibly different from h), and $\Delta\theta(x) = \Theta_\tau(x)$, then:

- (1) $\Delta\Theta_\tau$ is periodic with period τ
- (2) $\sum_a^b \Theta_\tau(x) = \sum_{a+k\tau}^{b+k\tau} \Theta_\tau(x)$ for any $k \in \mathbb{Z}$
- (3) $\Delta\theta(x+k\tau) = \Theta_\tau(x)$ for any $k \in \mathbb{Z}$
- (4) $\Delta[\theta(x) - \theta(x+k\tau)] = 0$

In general, $\sum \Theta_\tau(x)$ need not be periodic (with period τ). For example, the indefinite sum of $\tan x$ is $T(x)$, which is note periodic.

Odd and Even Functions If Φ is an even function, that is $\Phi(-x) = \Phi(x)$, and $\Delta\phi(x) = \Phi(x)$, then:

- (1) $\Delta\Phi(x)$ is odd around $x = -1/2$, that is $\Delta\Phi(u)|_{-x} = -\Delta\Phi(u)|_{x-1}$
- (2) $\sum_a^b \Phi(x) = \sum_{-a-1}^{-b-1} \Phi(x)$

$$(3) \Delta[-\phi(1-x)] = \Phi(x)$$

$$(4) \Delta[\phi(x) + \phi(1-x)] = 0$$

If Φ is an odd function, that is $\Phi(-x) = -\Phi(x)$, and $\Delta\phi(x) = \Phi(x)$, then:

$$(1) \Delta\Phi(x) \text{ is even around } x = -1/2, \text{ that is } \Delta\Phi(u)|_{-x} = \Delta\Phi(u)|_{x-1},$$

$$(2) \sum_a^b \Phi(x) = -\sum_{-a-1}^{-b-1} \Phi(x) = \sum_{-b-1}^{-a-1} \Phi(x)$$

$$(3) \Delta[\phi(1-x)] = \Phi(x)$$

$$(4) \Delta[\phi(x) - \phi(1-x)] = 0$$

Techniques of Boros and Moll [BOROS] If f and g satisfy the functional equation on the left, then the sum on the right is valid. The notation $f^{[n]}$ is used to denote the function f composed with itself n times.

$$g(x) = \nu f(\lambda(x)) - f(x) \quad \sum_{k=0}^{n-1} \nu^k g(\lambda^{[k]}(x)) = \nu^n f(\lambda^{[n]}(x)) - f(x) \quad (3.1.36)$$

$$g(x) = 2f\left(\frac{x}{2}\right) - f(x) \quad \sum_{k=0}^{n-1} 2^{-k-1} g(2^{k+1}x) = f(x) - 2^{-n} f(2^n x) \quad (3.1.37)$$

$$g(x) = \frac{f(x/m_2) - r_1 f(m_1 x/m_2)}{r_2} \quad \sum_{k=0}^{n-1} r_1^k g(m_1^k x) = \frac{f(x/m_2) - r_1^n f(m_1^n x/m_2)}{r_2} \quad (3.1.38)$$

3.2. Differences

3.2.1. Powers

$$\Delta c = 0 \quad (3.2.1)$$

$$\Delta ax = ah \quad (3.2.2)$$

$$\Delta x^2 = 2hx + h^2 \quad (3.2.3)$$

$$\Delta x^3 = 3hx^2 + 3h^2x + h^3 \quad (3.2.4)$$

$$\Delta x^4 = 4hx^3 + 6h^2x^2 + 4h^3x + h^4 \quad (3.2.5)$$

$$\Delta x^n = \sum_{k=0}^{n-1} \binom{n}{k} x^k h^{n-k} \quad (3.2.6)$$

$$\Delta x^{\langle a \rangle_h} = ax^{\langle a-1 \rangle_h} \quad (3.2.7)$$

3.2.2. Exponential Functions

$$\Delta a^x = a^x (a^h - 1) \quad (3.2.8)$$

$$\Delta \log_a x = \log_a \left(1 + \frac{h}{x} \right) \quad (3.2.9)$$

$$\Delta a^{\langle x \rangle} = a^{\langle x \rangle} (x + 1 - a) \quad h = 1 \quad (3.2.10)$$

$$\Delta \mathcal{F}(x) = \mathcal{F}(x - 1) \quad h = 1 \quad (3.2.11)$$

3.2.3. Trigonometric Functions

$$\Delta \sin ax = 2 \sin \frac{ah}{2} \cos \left(ax + \frac{ah}{2} \right) \quad (3.2.12)$$

$$\Delta \cos ax = -2 \sin \frac{ah}{2} \sin \left(ax + \frac{ah}{2} \right) \quad (3.2.13)$$

$$\Delta \tan ax = \frac{\sin ah}{\cos(ax + ah) \cos ax} \quad (3.2.14)$$

$$\Delta \cot ax = \frac{\sin ah}{\sin(ax + ah) \sin ax} \quad (3.2.15)$$

$$\Delta \sec ax = \frac{\sin(ax + \frac{ah}{2}) \sin(\frac{ah}{2})}{\cos(ax + ah) \cos ax} \quad (3.2.16)$$

$$\Delta \csc ax = -\frac{\cos(ax + \frac{ah}{2}) \sin(\frac{ah}{2})}{\sin(ax + ah) \sin ax} \quad (3.2.17)$$

$$\Delta \sin^2 x = \sin h \sin(2x + h) \quad (3.2.18)$$

$$\Delta \cos^2 x = -\sin h \sin(2x + h) \quad (3.2.19)$$

$$\Delta \tan^2 x = \frac{\sin h \sin(2x + h)}{\cos^2 x \cos^2(x + h)} \quad (3.2.20)$$

$$\Delta \cot^2 x = -\frac{\sin h \sin(2x + h)}{\sin^2 x \sin^2(x + h)} \quad (3.2.21)$$

3.2.4. Inverse Trigonometric Functions

$$\Delta \asin ax = \asin \left(a(x + h) \sqrt{1 - a^2 x^2} - ax \sqrt{1 - a^2 (x + h)^2} \right) \quad (3.2.22)$$

$$\Delta \acos ax = \acos \left(a^2 x(x + h) - \sqrt{(1 - a^2 (x + h)^2)(1 - a^2 x^2)} \right) \quad (3.2.23)$$

$$\Delta \atan ax = \atan \left(\frac{ah}{a^2 x^2 + a^2 xh + 1} \right) \quad (3.2.24)$$

$$\Delta \acot ax = -\acot \left(\frac{a^2 x^2 + a^2 xh + 1}{ah} \right) \quad (3.2.25)$$

$$\Delta \asec ax = -\asec \left(\frac{a^2 x(x + h)}{1 - \sqrt{(a^2 x^2 - 1)(a^2 (x + h)^2 - 1)}} \right) \quad (3.2.26)$$

$$\Delta \acsc ax = \asec \left(\frac{a^2 x(x + h) \sqrt{(1 - a^2 x^2)(1 - a^2 (x + h)^2)}}{ax \sqrt{1 - a^2 (x + h)^2} + a(x + h) \sqrt{1 - a^2 x^2}} \right) \quad (3.2.27)$$

3.2.5. Hyperbolic Functions

$$\Delta \sinh x = 2 \cosh\left(x + \frac{h}{2}\right) \sinh\left(\frac{h}{2}\right) \quad (3.2.28)$$

$$\Delta \cosh x = 2 \sinh\left(x + \frac{h}{2}\right) \sinh\left(\frac{h}{2}\right) \quad (3.2.29)$$

$$\Delta \tanh x = \frac{\sinh h}{\cosh(x+h) \cosh(x)} \quad (3.2.30)$$

$$\Delta \coth x = -\frac{\sinh h}{\sinh(x+h) \sinh(x)} \quad (3.2.31)$$

$$\Delta \operatorname{sech} x = -\frac{2 \sinh(x + \frac{h}{2}) \sinh(\frac{h}{2})}{\cosh(x+h) \cosh(x)} \quad (3.2.32)$$

$$\Delta \operatorname{csch} x = -\frac{2 \cosh(x + \frac{h}{2}) \sinh(\frac{h}{2})}{\sinh(x+h) \sinh(x)} \quad (3.2.33)$$

3.2.6. Exponential Sums

$$\Delta \operatorname{Bs}(x) = \frac{2^x}{x} \quad (3.2.34)$$

$$\Delta \operatorname{Ss}(x) = \frac{\sin x}{x} \quad (3.2.35)$$

$$\Delta \operatorname{Cs}(x) = \frac{\cos x}{x} \quad (3.2.36)$$

$$\Delta \operatorname{Sds}(x) = \frac{\operatorname{sind} x}{x} \quad (3.2.37)$$

$$\Delta \operatorname{Cds}(x) = \frac{\operatorname{cosd} x}{x} \quad (3.2.38)$$

3.2.7. Discrete Additive Trigonometric Functions

$$\Delta \operatorname{sind} x = \operatorname{cosd} x \quad (3.2.39)$$

$$\Delta \operatorname{cosd} x = -\operatorname{sind} x \quad (3.2.40)$$

3.2.8. Gamma and Related Functions

Here, $h = 1$.

$$\Delta\Gamma(x) = (x - 1)\Gamma(x) \quad (3.2.41)$$

$$\Delta\Gamma^{(-1)}(x) = \Gamma(x) \quad (3.2.42)$$

$$\Delta x\Gamma^{(-1)}(x) = \Gamma^{(-1)}(x + 2) \quad (3.2.43)$$

$$\Delta \frac{a^{(x)}}{\Gamma(x + k)} = \frac{a^{(x)}(a - k)}{\Gamma(x + k + 1)} \quad (3.2.44)$$

$$\Delta B(x, a) = \frac{-a}{x + a} B(x, a) \quad (3.2.45)$$

$$\Delta\psi(x) = \frac{1}{x} \quad (3.2.46)$$

$$\Delta\psi(x) = -\frac{1}{x^2} \quad (3.2.47)$$

$$\Delta\psi^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \quad (3.2.48)$$

$$\Delta\Gamma_q(x) = \frac{q - q^x}{1 - q} \Gamma_q(x) \quad 0 < q < 1 \quad (3.2.49)$$

$$\Delta P(x, a) = \frac{-a^x e^{-a}}{\Gamma(x + 1)} \quad (3.2.50)$$

$$\Delta\gamma(x, a) = (x - 1)\gamma(x, a) - a^x e^a \quad (3.2.51)$$

$$\Delta\Gamma(x, a) = (x - 1)\Gamma(x, a) + a^x e^a \quad (3.2.52)$$

$$\Delta\gamma^*(x, a) = \gamma^*(x, a) - \frac{1}{e^a \Gamma(x + 1)} \quad (3.2.53)$$

$$\Delta I_t(x, a) = (t - 1)I_t(x, a) + (t - 1)I_t(x, a - 1) \quad (3.2.54)$$

$$\Delta \binom{x}{a} = \binom{x}{a - 1} \quad (3.2.55)$$

$$\Delta \binom{-x}{a} = \binom{-x - 1}{a - 1} \quad (3.2.56)$$

3.2.9. More Exponential Forms

$$\Delta 2^x \frac{a^{2^x} + 1}{a^{2^x} - 1} = 2^x \frac{a^{2^x} - 1}{a^{2^x} + 1} \quad (3.2.57)$$

3.2.10. The Discrete Square Root

$$\Delta x^{\langle \frac{1}{2} \rangle} = \frac{1}{2} x^{\langle -\frac{1}{2} \rangle} = \frac{1}{2(x + \frac{1}{2})^{\langle \frac{1}{2} \rangle}} \quad (3.2.58)$$

$$\Delta (x^{\langle \frac{1}{2} \rangle})^2 = (x^{\langle -\frac{1}{2} \rangle})^2 \left(x + \frac{3}{4} \right) \quad (3.2.59)$$

$$\Delta (x^{\langle \frac{1}{2} \rangle})^3 = (x^{\langle -\frac{1}{2} \rangle})^3 \left(\frac{3}{2}x^2 + \frac{9}{4}x + \frac{7}{8} \right) \quad (3.2.60)$$

$$\Delta (x^{\langle \frac{1}{2} \rangle})^4 = (x^{\langle -\frac{1}{2} \rangle})^4 \left(2x^3 + \frac{7}{2}x^2 + \frac{9}{2}x + \frac{15}{16} \right) \quad (3.2.61)$$

$$\Delta (x^{\langle \frac{1}{2} \rangle})^n = (x^{\langle -\frac{1}{2} \rangle})^n \sum_{k=0}^{n-1} \binom{n}{k} \frac{2^{n-k}-1}{2^{n-k}} x^k \quad (3.2.62)$$

3.2.11. Forms involving trigonometric and exponential functions

$$\Delta 2^x \sin\left(\frac{a}{2^x}\right) = \sin\left(\frac{a}{2^{x+1}}\right) \sin^2\left(\frac{a}{2^{x+2}}\right) \quad (3.2.63)$$

$$\Delta 2^x \sin^2\left(\frac{a}{2^x}\right) = 2^{2x+2} \sin^4\left(\frac{a}{2^{x+1}}\right) \quad (3.2.64)$$

$$\Delta \tan\left(\frac{a}{2^x}\right) = -\frac{\tan\left(\frac{a}{2^{x+1}}\right)}{\cos\left(\frac{a}{2^x}\right)} \quad (3.2.65)$$

$$\Delta(-2)^x \sin\left(\frac{a}{2^x}\right) = (-2)^{x+3} \sin\left(\frac{a}{2^{x+1}}\right) \cos\left(\frac{a}{2^{x+1}}\right) \quad (3.2.66)$$

$$\Delta \frac{1}{2^x \sin\left(\frac{a}{2^x}\right)} = -2 \frac{\sin^2\left(\frac{a}{2^{x+2}}\right)}{2^x \sin\left(\frac{a}{2^x}\right)} \quad (3.2.67)$$

$$\Delta \frac{1}{2^x \tan\left(\frac{a}{2^x}\right)} = \frac{\tan\left(\frac{a}{2^{x+1}}\right)}{2^{x+1}} \quad (3.2.68)$$

$$\Delta \left(\frac{1}{2^x \tan\left(\frac{a}{2^x}\right)} \right)^2 = \frac{2}{2^x} - \left(\frac{\tan\left(\frac{a}{2^{x+1}}\right)}{2^{x+1}} \right) \quad (3.2.69)$$

$$\Delta \frac{\cos\left(\frac{a}{2^x}\right)}{2^x \sin\left(\frac{a}{2^x}\right)} = \frac{\sin\left(\frac{a}{2^x}\right)}{(2^{x+1} \cos\left(\frac{a}{2^{x+1}}\right))} \quad (3.2.70)$$

$$\Delta \cot(2^x a) = -\frac{1}{\sin(2^{x+1} a)} \quad (3.2.71)$$

$$\Delta \frac{\ln(2 \sin 2^x a)}{2^x} = -\frac{\log \tan 2^x a}{2^{x+1}} \quad (3.2.72)$$

$$\Delta \frac{1}{(2^x \sin\left(\frac{a}{2^x}\right))^2} = -\frac{1}{(2^{x+1} \cos\left(\frac{a}{2^{x+1}}\right))^2} \quad (3.2.73)$$

$$\Delta \frac{1}{\prod_{k=0}^{2n+1} \cos(a(x+k))} = 2 \sin(a(n+1)) \frac{\sin(a(x+n+1))}{\prod_{k=0}^{2n+2} \cos(a(x+k))} \quad (3.2.74)$$

$$\Delta \frac{(-1)^x}{\sin(ax) \sin(a(x+1))} = 2 \cos a \frac{(-1)^{x+1}}{\sin(ax) \sin(a(x+2))} \quad (3.2.75)$$

$$\Delta \frac{(-1)^x}{\prod_{k=0}^{2n+1} \sin(a(x+k))} = 2 \cos(a(n+1)) \frac{(-1)^{x+1} \sin(a(x+n+1))}{\prod_{k=0}^{2n+2} \sin(a(x+k))} \quad (3.2.76)$$

$$\Delta \frac{(-1)^x}{\cos(ax) \cos(a(x+1))} = 2 \cos a \frac{(-1)^{x+1}}{\cos(ax) \cos(a(x+2))} \quad (3.2.77)$$

$$\Delta \frac{(-1)^x}{\prod_{k=0}^{2n+1} \cos(a(x+k))} = 2 \cos(a(n+1)) \frac{(-1)^{x+1} \cos(a(x+2n+1))}{\prod_{k=0}^{2n+2} \cos(a(x+k))} \quad (3.2.78)$$

3.2.12. Forms involving the inverse tangent function

$$\Delta \operatorname{atan}\left(\frac{ax+b}{cx+d}\right) = \operatorname{atan}\left(\frac{bc-ad}{a^2 + ab + c^2 + cd + (2ab + b^2 + 2cd + d^2)x + (b^2 + d^2)x^2}\right) \quad (3.2.79)$$

$$\Delta \operatorname{atan}(xf(x)) = \operatorname{atan}\left(\frac{\Delta f(x)}{1 + f(x)f(x+1)}\right) \quad (3.2.80)$$

$$\Delta 2^x \operatorname{atan}\left(\frac{a}{2^x}\right) = 2^x \operatorname{atan}\left(\frac{a^3}{2^{3x+2} + 3a^2 2^x}\right) \quad (3.2.81)$$

3.3. Sums

3.3.1. Basics

$$\sum a = ahx + C \quad (3.3.1)$$

$$\sum (-1)^x \binom{n}{x} = (-1)^{x+1} \binom{n-1}{x-1} + C \quad (3.3.2)$$

$$\sum x = \frac{1}{2}x^2 - \frac{h}{2}x + C \quad (3.3.3)$$

$$\sum x^2 = \frac{1}{3}x^3 - \frac{2h}{3}x^2 - \frac{h^2}{6}x + C \quad (3.3.4)$$

$$\sum x^3 = \frac{1}{4}x^4 - \frac{h}{2}x^3 + \frac{h^2}{4}x^2 + C \quad (3.3.5)$$

$$\sum x^4 = \frac{1}{5}x^5 - \frac{h}{2}x^4 + \frac{h^2}{3}x^3 - \frac{h^4}{30}x + C \quad (3.3.6)$$

$$\sum x^n = \sum_{k=1}^n s(n, k) \frac{x^{\langle k+1 \rangle}}{k+1} + C \quad h = 1 \quad (3.3.7)$$

$$\sum x^n = \frac{B_{n+1}(x)}{n+1} + C \quad h = 1 \quad (3.3.8)$$

$$\sum x^n = \frac{E_n(x)}{2} + C \quad h = 1 \quad (3.3.9)$$

$$\sum x^{\langle a \rangle} = \frac{x^{\langle a+1 \rangle}}{a+1} + C \quad h = 1, a \neq -1 \quad (3.3.10)$$

$$\sum \frac{1}{x} = \psi(x) + C \quad h = 1 \quad (3.3.11)$$

$$\sum \frac{1}{x^n} = \frac{(-1)^{n-1} \psi^{(n-1)}(x)}{(n-1)!} + C \quad h = 1 \quad (3.3.12)$$

$$\sum \frac{1}{x^2 - 1} = \frac{1-2x}{2x(x-1)} \quad h = 1 \quad (3.3.13)$$

$$\sum a^x = \frac{h}{a^h - 1} a^x + C \quad (3.3.14)$$

$$\sum x a^x = \frac{(1-x)a^x}{1-a} - \frac{a^x}{(1-a)^2} \quad h = 1, a \neq 1 \quad (3.3.15)$$

$$\sum (x^2 a^x) = \frac{(-a^2 + 2x - 1)a^{x+2} + (2x^2 - 2x - 1)ax + 1 - x^2 a^x + x^2 + a}{(1-a)^3} \quad h = 1, a \neq 1 \quad (3.3.16)$$

$$\sum \frac{2^x}{x} = \text{Bs}(x) + C \quad h = 1 \quad (3.3.17)$$

$$\sum \log_a b x = \log_a \Gamma(x) + (\log_a b)x + C \quad h = 1 \quad (3.3.18)$$

$$\sum \log_a b x^n = n \log_a \Gamma(x) + (\log_a b)x + C \quad h = 1 \quad (3.3.19)$$

3.3.2. Trigonometric Functions

For all of the sums below, $h = 1$.

$$\sum \sin(2ax) = -\frac{\cos[a(2x-1)]}{2 \sin a} + C \quad a \neq \pi n \quad (3.3.20)$$

$$\sum \sin(2\pi nx) = x \sin(2\pi nx) + C \quad (3.3.21)$$

$$\sum \cos(2ax) = \frac{\sin[a(2x-1)]}{2 \sin a} + C \quad a \neq \pi n \quad (3.3.22)$$

$$\sum \cos(2\pi nx) = x \cos(2\pi nx) + C \quad (3.3.23)$$

$$\sum \sin^2(ax) = \frac{x}{2} - \frac{\sin[a(2x-1)]}{4 \sin a} + C \quad a \neq \pi n \quad (3.3.24)$$

$$\sum \sin^2(\pi nx) = x \sin^2(2\pi nx) + C \quad (3.3.25)$$

$$\sum \cos^2(ax) = \frac{x}{2} + \frac{\sin[a(2x-1)]}{4 \sin a} + C \quad a \neq \pi n \quad (3.3.26)$$

$$\sum \cos^2(\pi nx) = x \cos^2(2\pi nx) + C \quad (3.3.27)$$

$$\sum x \sin(2ax) = \frac{\sin[a(2x-1)]}{4 \sin^2 a} - x \frac{\cos[a(2x-1)]}{2 \sin a} + C \quad a \neq \pi n \quad (3.3.28)$$

$$\sum x \sin(2\pi nx) = \frac{1}{2}x(x-1) \sin^2(2\pi nx) + C \quad (3.3.29)$$

$$\sum x \cos(2ax) = \frac{\cos[a(2x-1)]}{4 \sin^2 a} + x \frac{\sin[a(2x-1)]}{2 \sin a} + C \quad a \neq \pi n \quad (3.3.30)$$

$$\sum x \cos(2\pi nx) = \frac{1}{2}x(x-1) \cos^2(2\pi nx) + C \quad (3.3.31)$$

$$\sum (-1)^x \cos(2bx) = (-1)^{x+1} \frac{\cos[b(2x-1)]}{2 \cos b} \quad b \neq \pi \left(n + \frac{1}{2}\right) + C \quad (3.3.32)$$

$$\sum (-1)^x \cos \left(2\pi \left(n + \frac{1}{2}\right)x\right) = (-1)^x x \cos \left(2\pi \left(n + \frac{1}{2}\right)x\right) + C \quad (3.3.33)$$

$$\sum (-1)^x \sin(2bx) = (-1)^{x+1} \frac{\sin[b(2x-1)]}{2 \cos b} \quad b \neq \pi \left(n + \frac{1}{2}\right) + C \quad (3.3.34)$$

$$\sum (-1)^x \sin \left(2\pi \left(n + \frac{1}{2}\right)x\right) = (-1)^x x \sin \left(2\pi \left(n + \frac{1}{2}\right)x\right) + C \quad (3.3.35)$$

$$\sum a^x \sin(bx) = a^x \frac{a \sin[b(x-1)] - \sin(bx)}{a^2 - 2a \cos b + 1} + C \quad a > 0, \quad a \neq 1 \quad (3.3.36)$$

$$\sum a^x \cos(bx) = a^x \frac{a \cos[b(x-1)] - \cos(bx)}{a^2 - 2a \cos b + 1} + C \quad a > 0, \quad a \neq 1 \quad (3.3.37)$$

$$\sum \frac{\sin x}{x} = \text{Ss}(x) + C \quad (3.3.38)$$

$$\sum \frac{\cos x}{x} = \text{Cs}(x) + C \quad (3.3.39)$$

$$\sum \tan \pi x = x \tan \pi x + C \quad (3.3.40)$$

$$\sum \cot \pi x = x \cot \pi x + C \quad (3.3.41)$$

$$\sum \tan ax = T_a(x) + C \quad (3.3.42)$$

$$\sum \cot ax = T_a\left(x + \frac{\pi}{2a}\right) + C \quad (3.3.43)$$

$$\sum \sec x = \frac{1}{2}T\left(\frac{x + \pi/2}{2}\right) + \frac{1}{2}T\left(\frac{x + 1 + \pi/2}{2}\right) - \frac{1}{2}T\left(\frac{x - \pi/2}{2}\right) - \frac{1}{2}T\left(\frac{x + 1 - \pi/2}{2}\right) + C \quad (3.3.44)$$

$$\sum \csc x = \frac{1}{2}T\left(\frac{x}{2}\right) + \frac{1}{2}T\left(\frac{x+1}{2}\right) - \frac{1}{2}T\left(\frac{x+\pi}{2}\right) - \frac{1}{2}T\left(\frac{x+1+\pi}{2}\right) + C \quad (3.3.45)$$

3.3.3. Hyperbolic Functions

For all of the sums below, $h = 1$.

$$\sum \sinh(2ax) = \frac{\cosh[a(2x-1)]}{2 \sinh(a)} + C \quad a \neq \pi n \quad (3.3.46)$$

$$\sum \sinh(2\pi nx) = x \sinh(2\pi nx) + C \quad (3.3.47)$$

$$\sum \cosh(2ax) = \frac{\sinh[a(2x-1)]}{2 \sinh(a)} + C \quad a \neq \pi n \quad (3.3.48)$$

$$\sum \cosh(2\pi nx) = x \cosh(2\pi nx) + C \quad (3.3.49)$$

$$\sum \sinh^2(ax) = \frac{\sinh[a(2x-1)]}{4 \sinh(a)} - \frac{x}{2} + C \quad a \neq \pi n \quad (3.3.50)$$

$$\sum \sinh^2(\pi nx) = x \sinh^2(\pi nx) + C \quad (3.3.51)$$

$$\sum \cosh^2(ax) = \frac{\sinh[a(2x-1)]}{4 \sinh(a)} + \frac{x}{2} + C \quad a \neq \pi n \quad (3.3.52)$$

$$\sum \cosh^2(\pi nx) = x \cosh^2(\pi nx) + C \quad (3.3.53)$$

$$\sum x \sinh(2ax) = \frac{\sinh[a(2x-1)]}{4 \sinh^2 a} - x \frac{\cosh[a(2x-1)]}{2 \sinh a} + C \quad a \neq \pi n \quad (3.3.54)$$

$$\sum x \sinh(2\pi nx) = \frac{1}{2}x(x-1) \sinh^2(2\pi nx) + C \quad (3.3.55)$$

$$\sum x \cosh(2ax) = \frac{\cosh[a(2x-1)]}{4 \sinh^2 a} + x \frac{\sinh[a(2x-1)]}{2 \sinh a} + C \quad a \neq \pi n \quad (3.3.56)$$

$$\sum x \cosh(2\pi nx) = \frac{1}{2}x(x-1) \cosh^2(2\pi nx) + C \quad (3.3.57)$$

$$\sum (-1)^x \cosh(2bx) = (-1)^x \frac{\cosh[b(2x-1)]}{2 \cosh b} + C \quad (3.3.58)$$

$$\sum (-1)^x \sinh(2bx) = (-1)^{x+1} \frac{\sinh[b(2x-1)]}{2 \cosh b} + C \quad (3.3.59)$$

$$\sum a^x \sinh(bx) = a^x \frac{a \sinh[b(x-1)] - \sinh(bx)}{a^2 - 2a \cosh b + 1} + C \quad a > 0, \quad a \neq 1 \quad (3.3.60)$$

$$\sum a^x \cosh(bx) = a^x \frac{a \cosh[b(x-1)] - \cosh(bx)}{a^2 - 2a \cosh b + 1} + C \quad a > 0, \quad a \neq 1 \quad (3.3.61)$$

$$\sum \tanh \pi i x = (x - 1) \tanh \pi i x + C \quad (3.3.62)$$

$$\sum \coth \pi i x = (x - 1) \coth \pi i x + C \quad (3.3.63)$$

$$\sum \tanh x = 2\epsilon_i(x) + x + C \quad (3.3.64)$$

$$\sum \coth x = 2\epsilon_i(x - \pi i/2) - x + C \quad (3.3.65)$$

$$\sum \operatorname{sech} x = i\epsilon_i \left(\frac{x - \pi i/2}{2} \right) + i\epsilon_i \left(\frac{x + 1 - \pi i/2}{2} \right) - i\epsilon_i \left(\frac{x - \pi i/2}{2} \right) - i\epsilon_i \left(\frac{x + 1 - \pi i/2}{2} \right) + C \quad (3.3.66)$$

$$\sum \operatorname{csch} x = i\epsilon_i \left(\frac{x - \pi i}{2} \right) + i\epsilon_i \left(\frac{x + 1 - \pi}{2} \right) - i\epsilon_i \left(\frac{x}{2} \right) - i\epsilon_i \left(\frac{x + 1}{2} \right) + C \quad (3.3.67)$$

3.3.4. Discrete Trigonometric Functions

$$\sum \operatorname{sind} x = -\operatorname{cosd} x \quad (3.3.68)$$

$$\sum \operatorname{cosd} x = \operatorname{sind} x \quad (3.3.69)$$

$$\sum \frac{\operatorname{sind} x}{x} = \operatorname{Sds}(x) + C \quad (3.3.70)$$

$$\sum \frac{\operatorname{cosd} x}{x} = \operatorname{Cds}(x) + C \quad (3.3.71)$$

3.3.5. Exponential Sums

For all of the sums below, $h = 1$.

$$\sum \operatorname{Bs}(x) = (x - 1) \operatorname{Bs}(x - 1) - 2^{x-1} + C \quad (3.3.72)$$

$$\sum \operatorname{Ss}(x) = (x - 1) \operatorname{Ss}(x - 1) + \frac{\cos(x - \frac{3}{2})}{2 \sin \frac{1}{2}} + C \quad (3.3.73)$$

$$\sum \operatorname{Cs}(x) = (x - 1) \operatorname{Cs}(x - 1) - \frac{\sin(x - \frac{3}{2})}{2 \sin \frac{1}{2}} + C \quad (3.3.74)$$

$$\sum \operatorname{Sds}(x) = (x - 1) \operatorname{Sds}(x - 1) + \operatorname{cosd} x + C \quad (3.3.75)$$

$$\sum \operatorname{Cds}(x) = (x - 1) \operatorname{Cds}(x - 1) - \operatorname{sind} x + C \quad (3.3.76)$$

3.3.6. Gamma Functions

For all of the sums below, $h = 1$.

$$\sum \Gamma(x) = \Gamma^{(-1)}(x) = \frac{\Gamma(x)\Gamma(-x+1, -1)}{(-1)^{x-1}e} + C \quad (3.3.77)$$

$$\sum \frac{a^{\langle x \rangle}}{\Gamma(x+b)} = \frac{a^{\langle x \rangle}}{\Gamma(x+k+1)(a-1-k)} \quad (3.3.78)$$

$$\sum \Gamma^{(-1)}(x) = (x-2)\Gamma^{(-1)}(x-2) + C \quad (3.3.79)$$

$$\sum \psi(x) = (x-1)\psi(x) - x + C \quad (3.3.80)$$

$$\sum \psi(-x) = x\psi(-x) - x + C \quad (3.3.81)$$

$$\sum x\psi(x) = \frac{x^{\langle 2 \rangle}\psi(x)}{2} - \frac{(x+1)^{\langle 2 \rangle}}{4} + C \quad (3.3.82)$$

$$\sum x^{\langle n \rangle}\psi(x) = \frac{x^{\langle n+1 \rangle}\psi(x)}{n+1} - \frac{x^{\langle n+1 \rangle}}{(n+1)^2} - \frac{(x-1)^{\langle n \rangle}}{n(n+1)} + C \quad (3.3.83)$$

$$\sum \psi(mx) = (\ln m - 1)x - \frac{m-1}{2m} + \frac{1}{m} \sum_{k=0}^{m-1} \left(x - 1 + \frac{k}{m} \right) \psi \left(x - \frac{k}{m} \right) + C \quad (3.3.84)$$

$$\sum \psi^{(1)}(x) = (x-1)\psi^{(1)}(x) + \psi(x) + C \quad (3.3.85)$$

$$\sum \psi^{(n)}(x) = (x-1)\psi^{(n)}(x) + (-1)^{n-1}\psi^{(n-1)}(x) + C \quad (3.3.86)$$

$$\sum \frac{1}{\Gamma(x)} = -\frac{\gamma^*(x-1, 1)}{e} + C \quad (3.3.87)$$

$$\sum \frac{x}{\Gamma(x+2)} = \frac{1}{\Gamma(x+1)} + C \quad (3.3.88)$$

$$\sum \frac{x}{\Gamma(x)} = \frac{1}{\Gamma(x-1)} - \frac{2\gamma^*(x-1, 1)}{e} + C \quad (3.3.89)$$

$$\sum \binom{x}{a} = \sum \frac{\Gamma(x)}{\Gamma(a)\Gamma(x-a)} = \binom{x}{a-1} + C \quad (3.3.90)$$

3.3.7. Forms involving $ax + b$

For all of the sums below, $h = 1$.

$$\sum \frac{1}{ax+b} = \frac{1}{a}\psi \left(x + \frac{b}{a} \right) + C \quad (3.3.91)$$

$$\sum \frac{x}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\psi \left(x + \frac{b}{a} \right) + C \quad (3.3.92)$$

$$\sum \frac{x^2}{ax+b} = \frac{ax^2 - (2b+1)x}{2a^2} + \frac{b^2}{a^3}\psi \left(x + \frac{b}{a} \right) + C \quad (3.3.93)$$

$$\sum \frac{1}{x(ax+b)} = \frac{1}{b}\psi(x) - \frac{1}{b}\psi \left(x + \frac{b}{a} \right) + C \quad (3.3.94)$$

3.3.8. Forms involving the sum of squares

For all of the sums below, $h = 1$. In these identities, the right hand side is real-valued if x is real.

$$\sum \frac{1}{x^2 + a^2} = \frac{1}{2ai} (\psi(x - ia) - \psi(x + ia)) + C \quad (3.3.95)$$

$$\sum \frac{x}{x^2 + a^2} = \frac{1}{2} (\psi(x - ia) + \psi(x + ia)) + C \quad (3.3.96)$$

$$\sum \frac{x^2}{x^2 + a^2} = x - \frac{a}{2i} (\psi(x - ia) - \psi(x + ia)) + C \quad (3.3.97)$$

$$\sum \frac{x^3}{x^2 + a^2} = \frac{x^{(2)}}{2} - \frac{a^2}{2} (\psi(x - ia) + \psi(x + ia)) + C \quad (3.3.98)$$

$$\sum \frac{1}{x(x^2 + a^2)} = \frac{1}{a^2} \psi(x) - \frac{1}{2a^2} (\psi(x - ai) + \psi(x + ai)) + C \quad (3.3.99)$$

$$\sum \frac{1}{x^2(x^2 + a^2)} = -\psi^{(1)}(x) - \frac{1}{2ai} (\psi(x - ia) - \psi(x + ia)) + C \quad (3.3.100)$$

$$\sum \frac{1}{x^3(x^2 + a^2)} = \frac{1}{2} \psi^{(2)}(x) - \psi(x) + \frac{1}{4ai} (\psi(x - ia) - \psi(x + ia)) + C \quad (3.3.101)$$

$$\sum \frac{1}{(x^2 + a^2)^2} = \frac{i}{4a^3} \psi(x + ia) - \frac{1}{4a^3} \psi^{(1)}(x + ia) - \frac{i}{4a^3} \psi(x - ia) - \frac{1}{4a^3} \psi^{(1)}(x - ia) + C \quad (3.3.102)$$

$$\sum \frac{x}{(x^2 + a^2)^2} = \frac{i}{4a^2} \psi^{(1)}(x + ia) - \frac{i}{4a^2} \psi^{(1)}(x - ia) + C \quad (3.3.103)$$

3.3.9. Forms involving general quadratic expressions

In the following, we use

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2}$$

For all of the sums below, $h = 1$.

$$\sum \frac{1}{x^2 + bx + c} = \frac{1}{\beta - \alpha} \psi(x - \alpha) - \frac{1}{\beta - \alpha} \phi_0(x - \beta) + C \quad (3.3.104)$$

$$\sum \frac{x}{x^2 + bx + c} = \frac{\beta}{\alpha - \beta} \psi(x - \alpha) - \frac{\beta}{\alpha - \beta} \phi_0(x - \beta) + C \quad (3.3.105)$$

3.3.10. Discrete Square Root

For all of the sums below, $h = 1$.

$$\sum \frac{1}{x^{\langle \frac{1}{2} \rangle}} = 2 \left(x + \frac{1}{2} \right) + C \quad (3.3.106)$$

$$\sum \frac{x}{x^{\langle \frac{1}{2} \rangle}} = 2 \left(x + \frac{1}{2} \right)^{\langle \frac{3}{2} \rangle} - \frac{4}{15} x^{\langle \frac{5}{2} \rangle} + C \quad (3.3.107)$$

3.3.11. Forms involving the exponential function

Note, the function ψ_a is the q -digamma function (and not the poly-gamma function).

$$\sum \frac{1}{e^{2ix} + 1} = -\frac{1}{2} (iT(x) + x) + C \quad (3.3.108)$$

$$\sum \frac{1}{e^{2ix} + 1} = \sum E(2x) = \epsilon(x) + C \quad (3.3.109)$$

$$\sum \frac{1}{e^{ix} + 1} = -\frac{1}{2} \left(iT\left(\frac{x}{2}\right) + iT\left(\frac{x+1}{2}\right) + x + \frac{1}{2} \right) + C \quad (3.3.110)$$

$$\sum \frac{1}{e^{ix} + 1} = \sum E(x) = \epsilon\left(\frac{x}{2}\right) + \epsilon\left(\frac{x+1}{2}\right) + C \quad (3.3.111)$$

$$\sum \frac{1}{e^{-2x} + 1} = \sum E(2ix) = \epsilon_i(x) + C \quad (3.3.112)$$

$$\sum \frac{1}{e^{-x} + 1} = \sum E(ix) = \epsilon_i\left(\frac{x}{2}\right) + \epsilon_i\left(\frac{x+1}{2}\right) + C \quad (3.3.113)$$

$$\sum \frac{1}{1-a^x} = x - \frac{\psi_a(x)}{\ln a} + C \quad |a| < 1 \quad (3.3.114)$$

$$\sum \frac{a^x}{1-a^x} = \frac{\psi_a(x)}{\ln a} + C \quad |a| < 1 \quad (3.3.115)$$

$$\sum \frac{1}{1+a^x} = x - \frac{\psi_a(x + \pi i / \ln a)}{\ln a} + C \quad |a| < 1 \quad (3.3.116)$$

$$\sum \frac{a^x}{1-a^x} = \frac{\psi_a(x + \pi i / \ln a)}{\ln a} + C \quad |a| < 1 \quad (3.3.117)$$

3.4. Taylor Series

3.4.1. Definition

The formal discrete Taylor series of a function f is given by:

$$f^*(x-a) = \sum_{k=0}^{\infty} f^{\langle n \rangle}(a) \frac{(x-a)^{\langle n \rangle}}{n!}. \quad (3.4.1)$$

We write $f(x) \sim f^*(x)$ to denote that f^* is the formal Taylor series of f . Note that two functions which differ only by a periodic function of period 1 will have the same discrete Taylor series.

Example Let us find the discrete Taylor series of $f(x) = 2^x$ at $x = 0$. Since, $f^{\langle n \rangle}(x) = 2^x$, we have $f^{\langle n \rangle}(0) = 1$. Thus, the Taylor series is given by

$$2^x \sim \sum_{k=0}^{\infty} \frac{x^{\langle n \rangle}}{n!}$$

Graphs for the truncated series is given in Figure 3.1. From the graphs it looks like $f^*(x)$ only converges for $x > 0$.

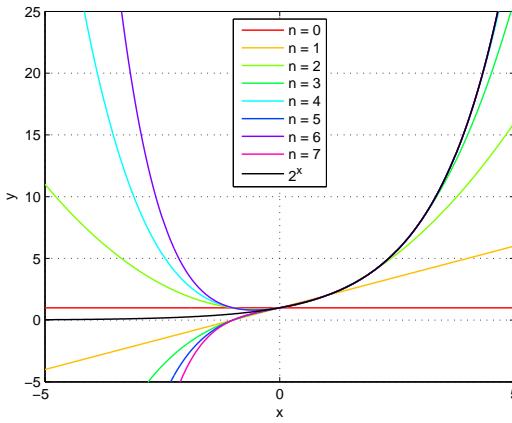


Figure 3.1.: Taylor series for 2^x

The following are valid for $x \in \mathbb{N}_0$.

3.4.2. Table

$$0 \sim \sum_{k=0}^{\infty} (-1)^k \frac{(x+1)^{\langle k \rangle}}{k!} \quad (3.4.2)$$

$$x^n \sim \sum_{k=0}^n S(n, k) x^{\langle k \rangle} \quad (3.4.3)$$

$$a^x \sim \sum_{k=0}^{\infty} (a-1)^k \frac{x^{\langle k \rangle}}{k!} \quad (3.4.4)$$

$$\log_a(x+1) \sim \sum_{k=0}^{\infty} \sum_{m=0}^k (-1)^{m+k} \binom{k}{m} \log_a(m+1) \frac{x^{\langle k \rangle}}{k!} \quad (3.4.5)$$

$$\sin(ax) \sim \sum_{k=0}^{\infty} 2^k \sin^k \left(\frac{a}{2} \right) \sin \left(\frac{(\pi+a)k}{2} \right) \frac{x^{\langle k \rangle}}{k!} \quad (3.4.6)$$

$$\cos(ax) \sim \sum_{k=0}^{\infty} 2^k \sin^k \left(\frac{a}{2} \right) \cos \left(\frac{(\pi+a)k}{2} \right) \frac{x^{\langle k \rangle}}{k!} \quad (3.4.7)$$

$$\text{sind}(x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)}}{(2k+1)!} \quad (3.4.8)$$

$$\text{cosd}(x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{x^{\langle 2k \rangle}}{(2k)!} \quad (3.4.9)$$

$$\frac{1}{x+1} \sim \sum_{k=0}^{\infty} (-1)^k \frac{x^{(k)}}{(k+1)!} \quad (3.4.10)$$

$$\psi(x+1) \sim \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{x^{(k)}}{k \cdot k!} \quad (3.4.11)$$

$$\Gamma(x+1) \sim \sum_{k=0}^{\infty} d(k+1) \frac{x^{(k)}}{k!} \quad (3.4.12)$$

$$\Gamma^{(-1)}(x+1) \sim \sum_{k=0}^{\infty} d(k) \frac{x^{(k)}}{k!} \quad (3.4.13)$$

$$(-1)^x d(x) = \sum_{k=0}^{\infty} (-1)^k x^{(k)} \quad (3.4.14)$$

$$(-1)^x d(x-1) = \sum_{k=0}^{\infty} (-1)^k k x^{(k)} \quad (3.4.15)$$

$$(-1)^x (x+2)d(x) + 1 = \sum_{k=0}^{\infty} (-1)^k (k+1) x^{(k)} \quad (3.4.16)$$

$$\text{Bs}(x+1) \sim \psi(x+1) + \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{x^{(k)}}{k!} \quad (3.4.17)$$

$$= \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k (k!)^2 + 1}{k \cdot k!} x^{(k)} \quad (3.4.18)$$

$$\text{Sds}(x+1) \sim \psi(x+1) + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k+1} \cdot \frac{x^{(2k+1)}}{(2k+1)!} \quad (3.4.19)$$

$$= \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{(2k+1) \cdot (2k+1)! (k-1)! + 1}{(2k+1) \cdot (2k+1)!} x^{(k)} \quad (3.4.20)$$

$$\text{Cds}(x+1) \sim \psi(x+1) + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k} \cdot \frac{x^{(2k)}}{(2k)!} \quad (3.4.21)$$

$$= \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{2k \cdot (2k)! (k-1)! + 1}{2k \cdot (2k)!} x^{(k)} \quad (3.4.22)$$

$$(3.4.23)$$

3.5. Analogs

3.5.1. Definition

If f is a function with Taylor expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

then the analog of F of f (in arithmetic difference calculus), is the function with discrete Taylor expansion

$$F(x) = \sum_{k=0}^{\infty} a_k x^{(k)}.$$

Example Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ and } 2^x = \sum_{k=0}^{\infty} \frac{x^{\langle k \rangle}}{k!},$$

2^x is the analog of e^x .

3.5.2. Properties

In the table below, F is the analog of f , and G is the analog of g , a and b are constants, and n is a constant integer.

Standard function	Analog ($h = 1$)	Analog ($h \neq 0$)
$af + bg$	$aF + bG$	$aF + bG$
Df	ΔF	ΔF
$\int f$	$\sum f$	$\sum f$
$xf(x)$	$xF(x - 1)$	$xF(x - h)$
$x^n f(x)$	$x^{\langle n \rangle} F(x - n)$	$x^{\langle n \rangle_h} F(x - hn)$

3.5.3. Table of Analogs

Below $\Theta(x)$ is *any* periodic function with period h .

Standard Function	Analog ($h = 1$)	Analog ($h \neq 0$)
c	$\Theta(x)$	$\Theta\left(\frac{x}{h}\right)$
x^n , with $n \in \mathbb{N}$	$x^{\langle a \rangle}$	$x^{\langle a \rangle_h}$
\sqrt{x}	$x^{\langle \frac{1}{2} \rangle}$	$x^{\langle \frac{1}{2} \rangle_h}$
e^x	2^x	$(\sqrt[h]{2})^x$
$\sin x$	$\text{sind } x$	$\text{sind}\left(\frac{x}{h}\right)$
$\cos x$	$\text{cosd } x$	$\text{cosd}\left(\frac{x}{h}\right)$
$(x+1)^{-1}$	$(-1)^x d(x)$	$(-1)^{x/h} d(x/h)$
$(x+1)^{-2}$	$(-1)^x (x+2)d(x) + 1$	$(-1)^x/h (x/h+2)d(x/h) + 1$
$v(a)^{-x} \sin(\sin(a)x)$	$\sin ax$	$\sin(ax/h)$
$v(a)^{-x} \cos(\sin(a)x)$	$\cos ax$	$\cos(ax/h)$

$$v_c(a) = \left(\frac{\cos(\cos(a/2))}{w_c(a)} \right)^{2 \sin(a/2)} \quad v_s(a) = \left(\frac{\sin(\cos(a/2))}{w_s(a)} \right)^{2 \sin(a/2)} \quad (3.5.1)$$

$$w_c(a) = \sum_{k=0}^{\infty} \frac{1}{k!} \cos\left(\frac{(\pi+a)k}{2}\right) \quad w_s(a) = \sum_{k=0}^{\infty} \frac{1}{k!} \sin\left(\frac{(\pi+a)k}{2}\right) \quad (3.5.2)$$

3.6. Miscellaneous Difference Identities

$$\Delta \mathcal{F}(x) = \mathcal{F}(x - 1) \quad (3.6.1)$$

$$\Delta W(x) = \Delta \ln x - \Delta \ln W(x) \quad (3.6.2)$$

3.7. Difference Equations

3.7.1.

$$\Delta f(x) = f(x)$$

$$f(x) = 2^x$$

3.7.2.

$$p(x)f(x+1) - f(x) = q(x)$$

$$f(x) = \frac{\sum(q(x) \prod p(x))}{\prod p(x)}$$

3.7.3.

$$f(x)\Delta f(x) = 1$$

$$f(x) = \frac{c\mathcal{F}(x) - \mathcal{F}(x-1)}{c\mathcal{F}(x-1) - \mathcal{F}(x-2)}$$

4. Geometric Differences and Sums

4.1. Definitions and Properties

4.1.1. Definitions

$$P f(x) = f(xh) - f(x) \quad h \neq 0 \quad (4.1.1)$$

$$P P^{-1} f(x) = f(x) \quad (4.1.2)$$

Thus, $P^{-1} f(x)$ is any function $F(x)$ that satisfies $P F(x) = f(x)$.

4.1.2. Properties of Operators

Basic Properties of Differencing

$$P(af(x) + bg(x)) = a P f(x) + b P g(x) \quad \text{Linearity} \quad (4.1.3)$$

$$P(f(x)g(x)) = f(hx) P g(x) + g(x) P f(x) \quad \text{Product Rule} \quad (4.1.4)$$

$$P \frac{f(x)}{g(x)} = \frac{g(x) P f(x) - f(x) P g(x)}{g(x)g(hx)} \quad \text{Quotient Rule} \quad (4.1.5)$$

$$P^n f(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(h^k x) \quad \text{nth Difference} \quad (4.1.6)$$

$$P^n(f(x)g(x)) = \sum_{k=0}^n P^k f(x) P^{n-k} g(h^k x) \quad \text{Leibniz's Rule} \quad (4.1.7)$$

Basic Properties of Summation

$$P^{-1}(af(x) + bg(x)) = a P^{-1} f(x) + b P^{-1} g(x) \quad \text{Linearity} \quad (4.1.8)$$

$$\sum_a^{b-1} f(h^{x-a} x) = P^{-1} f(x) \Big|_a^b \quad \text{Fundamental Theorem} \quad (4.1.9)$$

$$P^{-1} f(x)g(x) = f(x) P^{-1} g(x) - P^{-1} P f(x) P^{-1} g(x+1) \quad \text{Summation by parts} \quad (4.1.10)$$

$$(4.1.11)$$

Relationship with Arithmetic Difference Operators

$$P_x f(\log_h x) = \Delta_u f(u) \Big|_{u=\log_h x} \quad (4.1.12)$$

Chain rules

$$P_x f(ax) = P_u f(u) \Big|_{u=ax} \quad (4.1.13)$$

$$P_x f(x^m) = \sum_{k=0}^{m-1} P_u f(h^k u) \Big|_{u=x^m} \quad m \in \mathbb{N} \quad (4.1.14)$$

Substitution Rules

$$P_x^{-1} f(ax) = P_u^{-1} f(u)|_{u=ax} \quad (4.1.15)$$

$$P_u^{-1} f(\sqrt[m]{u}) = \sum_{k=0}^{m-1} (P_x^{-1} f(x))|_{x=\sqrt[m]{uh^k}} \quad m \in \mathbb{N} \quad (4.1.16)$$

4.2. Differences
4.2.1. Basics

$$P c = 0 \quad (4.2.1)$$

$$P x^a = (h^a - 1)x^a \quad (4.2.2)$$

$$P \log_a x = \log_a h \quad (4.2.3)$$

$$P(\log_h x)^{(n)} = n(\log_h x)^{(n-1)} \quad (4.2.4)$$

$$P a^x = a^x(a^{(h-1)x} - 1) \quad (4.2.5)$$

$$P 2^{\log_h x} = 2^{\log_h x} \quad (4.2.6)$$

4.3. Sums
4.3.1. Basics

$$P^{-1} a = a \log_h x + C \quad (4.3.1)$$

$$P^{-1} x^a = \frac{x^a}{h^a - 1} \quad (4.3.2)$$

$$P^{-1} \log_a x = \frac{\log_a x (\log_h x - 1)}{2} + C \quad (4.3.3)$$

$$P^{-1}(\log_h x)^{(n)} = \frac{(\log_h x)^{(n+1)}}{n+1} + C \quad (4.3.4)$$

$$P^{-1}(\log_a x)^{(n)} = \frac{\log_a x (\log_h x - 1)^{(n)}}{n+1} + C \quad (4.3.5)$$

4.3.2. Trigonometric Functions

$$P^{-1} \sin(x) = \sum_{k=1}^{\infty} \sin(h^{-k} x) = S(x, h) + C \quad (4.3.6)$$

$$P^{-1} \cos(x) = \log_h x + \sum_{k=1}^{\infty} (\cos(h^{-k} x) - 1) = C(x, h) + C \quad (4.3.7)$$

$$P^{-1} x \cos(x) = x S'(x, h) + C \quad (4.3.8)$$

$$P^{-1} x \sin(x) = -x C'(x, h) + C \quad (4.3.9)$$

$$(4.3.10)$$

4.4. Analogs

4.4.1. Properties

In the table below, F is the analog of f , and G is the analog of g , a and b are constants, and n is a constant integer.

Standard function	Analog
$af + bg$	$aF + bG$
Df	$P F$
$\int f$	$P^{-1} f$
$xf(x)$	$\log_h x F\left(\frac{x}{h}\right)$
$x^n f(x)$	$(\log_h x)^{(n)} F\left(\frac{x}{h}\right)$

4.4.2. Table of Analogs

Below $\Theta(x)$ is *any* periodic function with period h .

Elementary Function	Analog
c	$\Theta(\log_h x)$
x^a	$(\log_h x)^{(a)}$
\sqrt{x}	$(\log_h x)^{(\frac{1}{2})}$
e^x	$2^{\log_h x} = x^{\log_h 2}$
$\sin x$	$\text{sind } \log_h x$
$\cos x$	$\text{cosd } \log_h x$

5. Arithmetic Quotients and Products

5.1. Definitions and Properties

5.1.1. Definition

The quotient operator is defined as:

$$Q f(x) = \frac{f(x+1)}{f(x)} \quad (5.1.1)$$

The inverse operator is the indefinite product \prod :

$$Q \prod f(x) = f(x) \quad (5.1.2)$$

5.1.2. Properties of Operators

$$Q(f(x)^a g(x)^b) = (Q f(x))^a (Q g(x))^b \quad (5.1.3)$$

$$Q f(x)^{g(x)} = [Q e^{g(x)}]^{\ln f(x+1)} [Q f(x)]^{g(x)} \quad (5.1.4)$$

$$Q f(x)^{g(x)} = [Q e^{g(x)}]^{\ln f(x)} [Q f(x)]^{g(x+1)} \quad (5.1.5)$$

$$\prod_a^{b-1} f(x) = \frac{\prod f(x)|_{x=b}}{\prod f(x)|_{x=a}} \quad (5.1.6)$$

$$\prod f(x)^{g(x)} = \frac{(\prod f(x))^{g(x)}}{\prod (\prod f(x+1))^{\Delta g(x)}} \quad (5.1.7)$$

5.1.3. Relation to Additive Operators

$$a^{\Delta f(x)} = Q a^{f(x)} \quad (5.1.8)$$

$$a^{\sum f(x)} = \prod a^{f(x)} \quad (5.1.9)$$

5.2. Quotients

5.2.1. Basics

$$Q c = 1 \quad (5.2.1)$$

$$Q x = 1 + \frac{1}{x} \quad (5.2.2)$$

$$Q x^2 = 1 + \frac{2}{x} + \frac{1}{x^2} \quad (5.2.3)$$

$$Q x^n = \sum_{k=0}^n \binom{n}{k} \frac{1}{x^k} \quad (5.2.4)$$

$$Q x^{\langle a \rangle} = \frac{x+1}{x-a+1} \quad (5.2.5)$$

$$Q a^x = a \quad (5.2.6)$$

$$Q a^{2^x} = a^{2^x} \quad (5.2.7)$$

$$Q a^{b^x} = a^{b^x(b-1)} \quad (5.2.8)$$

$$Q \log_a x = \log_x(x+1) \quad (5.2.9)$$

$$Q \exp\left(x^{\langle n \rangle}\right) = \exp\left(nx^{\langle n-1 \rangle}\right) \quad (5.2.10)$$

5.2.2. Trigonometric Functions

$$Q \sin ax = \cos a + \sin a \cot ax \quad (5.2.11)$$

$$Q \cos ax = \cos a - \sin a \tan ax \quad (5.2.12)$$

$$Q \tan ax = \frac{\cos a + \sin a \cot ax}{\cos a - \sin a \tan ax} \quad (5.2.13)$$

$$Q \cot ax = \frac{\cos a - \sin a \tan ax}{\cos a + \sin a \cot ax} \quad (5.2.14)$$

$$Q \sec ax = \frac{1}{\cos a - \sin a \tan ax} \quad (5.2.15)$$

$$Q \csc ax = \frac{1}{\cos a + \sin a \cot ax} \quad (5.2.16)$$

$$(5.2.17)$$

5.2.3. Hyperbolic Functions

$$Q \sinh ax = \cosh a + \sinh a \coth ax \quad (5.2.18)$$

$$Q \cosh ax = \cosh a + \sinh a \tanh ax \quad (5.2.19)$$

$$Q \tanh ax = \frac{\cosh a + \sinh a \coth ax}{\cosh a + \sinh a \tanh ax} \quad (5.2.20)$$

$$Q \coth ax = \frac{\cosh a + \sinh a \tanh ax}{\cosh a + \sinh a \coth ax} \quad (5.2.21)$$

$$Q \operatorname{sech} ax = \frac{1}{\cosh a + \sinh a \tanh ax} \quad (5.2.22)$$

$$Q \operatorname{csch} ax = \frac{1}{\cosh a + \sinh a \coth ax} \quad (5.2.23)$$

$$(5.2.24)$$

5.2.4. Discrete Additive Trigonometric Functions

$$Q \sin d x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cot \left(\frac{\pi x}{4} \right) \quad (5.2.25)$$

$$Q \cos d x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \tan \left(\frac{\pi x}{4} \right) \quad (5.2.26)$$

5.2.5. Gamma and Related Functions

$$Q \Gamma(x) = x \quad (5.2.27)$$

$$Q \Gamma^n(x) = x^n \quad (5.2.28)$$

$$Q(a^x \Gamma^n(x)) = ax^n \quad (5.2.29)$$

$$Q \Gamma(x+a) = x+a \quad (5.2.30)$$

$$Q a^{\psi(x)} = a^{1/x} \quad (5.2.31)$$

$$Q B(a, x) = \frac{1}{x+a} \left(\frac{x \Gamma(x) + \Gamma(a)}{\Gamma(x) + \Gamma(a)} \right) \quad (5.2.32)$$

5.3. Products

5.3.1. Basics

$$\prod (ax^n) = C a^x \Gamma^n(x) \quad (5.3.1)$$

$$\prod (ax+b) = C a^x \Gamma \left(x + \frac{b}{a} \right) \quad (5.3.2)$$

$$\prod a^x = C a^{x(x-1)/2} \quad (5.3.3)$$

$$\prod a^{1/x} = C a^{\psi(x)} \quad (5.3.4)$$

$$\prod a \frac{(x-p_1)(x-p_2) \cdots (x-p_m)}{(x-q_1)(x-q_2) \cdots (x-q_n)} = C a^x \frac{\Gamma(x-p_1)\Gamma(x-p_2) \cdots \Gamma(x-p_m)}{\Gamma(x-q_1)\Gamma(x-q_2) \cdots \Gamma(x-q_n)} \quad (5.3.5)$$

$$\prod a^{\frac{1}{x}} = C a^{\psi(x)} \quad (5.3.6)$$

The function G used here is the Barnes G -function.

$$\prod \Gamma(x) = C \frac{\Gamma(x)^{x-1}}{K(x)} = CG(c) \quad (5.3.7)$$

$$(5.3.8)$$

5.3.2. Miscellaneous

$$\prod x^x = CK(x) \quad (5.3.9)$$

$$\prod \text{sexp}_a = C \frac{(\text{sexp}_a(x))'}{\text{sexp}_a(x)(\ln a)^x} \quad (5.3.10)$$

5.4. Taylor Series

$$f(x) = \prod_{n=0}^{\infty} f^Q n(0) \exp\left(\frac{x^{(n)}}{n!}\right) \quad (5.4.1)$$

5.5. Miscellaneous Identities

$$Q\mathcal{F}(x) = k + Q\mathcal{F}(x - k) \quad (5.5.1)$$

5.6. Quotient Equations

The solution of the equation:

$$\Delta \frac{f(x+1)^{p(x)}}{f(x)} = q(x) \quad (5.6.1)$$

is given by

$$f(x) = \left(\prod q(x)^{\prod p(x)} \right)^{1/\prod p(x)}$$

5.7. Analogs

5.7.1. Analogs of Elementary Functions

Elementary Function	Analog
x^n	$\exp(x^{(n)})$
x^{-n}	$\exp(x^{(-n)})$
e^x	$\exp(2^x)$
$\sin x$	$\exp(\text{sind } x)$
$\cos x$	$\exp(\text{cosd } x)$

6. Geometric Quotients and Products

6.1. Definition

$$R f(x) = \frac{f(xh)}{f(x)}$$

6.2. Quotients

6.2.1. Basics

$$R c = 1 \tag{6.2.1}$$

$$R x = h \tag{6.2.2}$$

$$R x^n = h^n \tag{6.2.3}$$

$$R a^x = a^{x(h-1)} \tag{6.2.4}$$

$$R \log_a x = 1 + \log_x h \tag{6.2.5}$$

$$R \log_x a = 1 - \frac{\ln h}{\ln h + \ln x} \tag{6.2.6}$$

7. Generalised Operators

7.1. Definitions

$$\Delta_g f(x) = f(g(x)) - f(x) \quad (7.1.1)$$

$$Q_g f(x) = \frac{f(g(x))}{f(x)} \quad (7.1.2)$$

7.2. Theorems

$$\Delta_g (af_1(x) + bf_2(x)) = a\Delta_g f_1(x) + b\Delta_g f_2(x) \quad (7.2.1)$$

$$\Delta_g (f_1(x)f_2(x)) = f_1(g(x))\Delta_g f_2(x) + f_2(x)\Delta_g f_1(x) \quad (7.2.2)$$

$$\Delta_g (f_1(x)f_2(x)) = f_1(x)\Delta_g f_2(x) + f_2(g(x))\Delta_g f_1(x) \quad (7.2.3)$$

$$\sum_{k=0}^{n-1} \Delta_g f(g^k(x)) = f(g^n(x)) - f(x) \quad \text{Fundamental Theorem} \quad (7.2.4)$$

$$\left(\prod_g g'(x) \right) D\Delta_g f(x) = \Delta_g \left(f'(x) \prod_g g'(x) \right) \quad (7.2.5)$$

7.3. Standard Functions of Difference Operators

7.4. Definitions

$$I_\Delta g(x) = f(x) \Rightarrow f(g(x)) - f(x) = 1 \quad (7.4.1)$$

7.5. Theorems

$$\Delta_g \Theta(I_\Delta g(x)) = 0 \quad \text{for any } \Theta(x+1) = \Theta(x) \quad (7.5.1)$$

$$\Delta_g 2^{I_\Delta g(x)} = 2^{I_\Delta g(x)} \quad (7.5.2)$$

$$\Delta_g (I_\Delta g(x))^{\langle n \rangle} = n (I_\Delta g(x))^{\langle n-1 \rangle} \quad (7.5.3)$$

$$\Delta_g \sin d(I_\Delta g(x)) = \cos d(I_\Delta g(x)) \quad (7.5.4)$$

$$\Delta_g \cos d(I_\Delta g(x)) = -\sin d(I_\Delta g(x)) \quad (7.5.5)$$

$$(7.5.6)$$

$$p(x+1) = g(p(x)) \Rightarrow I_\Delta g(x) = p^{-1}(x) \quad (7.5.7)$$

$$p(x) = I_\Delta g(x) \Rightarrow f(g(x)) = f(p(p^{-1}x) + k)) \quad (7.5.8)$$

In the above, 7.5.7 provides a way of determining $I_{\Delta}g(x)$, and 7.5.8 provides a way to convert a functional equation—where all occurrences of f are in the form $f(g(x))$ —to a difference equation.

7.6. Analogous Functions

In the table below, F is the analog (in $\Delta_{g(x)}$) of f , and G is the analog of g , and a and b are constants. The notation g^{-n} is used to denote the inverse of the function g composed with itself n times.

Standard Function	Analog
$af + bg$	$aF + bG$
Df	$\Delta_{g(x)}F$
$\int f$	$\sum_{g(x)} F$
$xf(x)$	$I_{\Delta g}(x)F(g^{-1}(x))$
$x^n f(x)$	$I_{\Delta g}(x)F(g^{-n}(x))$

7.7. Table

Here, w is an arbitrary constant.

$g(x)$		$I_{\Delta}g(x)$
$x + h$		$\frac{x}{h}$
$ax + b$		$\log_a \left(\frac{x(a-1)}{b} + 1 \right)$
hx	$h > 0$	$\log_h x$
$-hx$	$h > 0$	$\frac{\ln x}{\pi i + \ln h}$
x^h ,	$h > 0$	$\log_h \log_w x$
x^{-h} ,	$h > 0$	$\frac{\ln \log_w x}{\pi i + \ln h}$
h^x		$\text{slog}_h(x)$
$h - x$		$\frac{\ln(h-2x)}{\pi i}$
$\frac{h}{x}$	$h \neq 0, 1$	$\frac{\ln \left(\frac{2 \ln x}{\ln h} - 1 \right)}{\pi i}$

8. Trigonometric Equations

$$f(\cos(x)) + f(\sin(x)) = 1 \Rightarrow f(x) = \frac{2}{\pi} \cos^{-1} x + C \quad (8.0.1)$$

$$f(\cos(x)) + f(\sin(x)) = f(x) \Rightarrow f(x) = 2^{\frac{2}{\pi} \cos^{-1} x} C \quad (8.0.2)$$

Part II.

Transforms

9. The z-Transform

9.1. Definition and Properties

Definition This section deals with the one-sided z -transform. The z -transform of a function f is the function F given by

$$F(z) = \sum_{x=0}^{\infty} z^{-x} f(x), \quad (9.1.1)$$

if the series converges. The values of z for which the series converges is called the *domain of convergence* (DOC).

The transform only depends on the function at non-negative integer values. Thus,

$$\mathcal{Z}[f(x)] = \mathcal{Z}\left[f(x) \sum_{k=-\infty}^{\infty} \Delta(x)\right] \quad \text{Sampling} \quad (9.1.2)$$

$$\mathcal{Z}[f(x)] = \mathcal{Z}[f(x)u(x)] \quad \text{Truncation} \quad (9.1.3)$$

$$\mathcal{Z}[f(x)] = \mathcal{Z}[f(x) + \Theta(x)] - \Theta(0), \quad (9.1.4)$$

where

$$\Delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \quad \text{Unit pulse function} \quad (9.1.5)$$

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \text{Unit step function} \quad (9.1.6)$$

and $\Theta(x)$ is any periodic function with period 1.

Basic Properties If $F(z) = \mathcal{Z}[f(x)]$ and $G(z) = \mathcal{Z}[g(x)]$, then

$$\mathcal{Z}[af(x) + bg(x)] = aF(z) + bG(z) \quad \text{Linearity} \quad (9.1.7)$$

$$\mathcal{Z}[f(x-y)] = z^{-y}F(z) \quad \text{Backward Shift} \quad (9.1.8)$$

$$\mathcal{Z}[f(x+y)] = z^y \left[F(z) - \sum_{k=0}^{y-1} f(k)z^{-k} \right] \quad \text{Forward Shift} \quad (9.1.9)$$

$$\mathcal{Z}[f(-x)] = F\left(\frac{1}{z}\right) \quad \text{Reflection} \quad (9.1.10)$$

$$f(x^*) = F^*(z^*) \quad \text{Conjugation} \quad (9.1.11)$$

$$\mathcal{Z}[f(x) * g(x)] = F(z)G(z) \quad \text{Convolution} \quad (9.1.12)$$

$$\mathcal{Z}[f(x)g(x)] = \frac{1}{2\pi i} \oint_C \frac{F(w)G(\frac{z}{w})}{z} dw \quad (9.1.13)$$

$$\sum_{x=0}^{\infty} f(x)g^*(x) = \frac{1}{2\pi i} \oint_C \frac{F(w)G(\frac{1}{w^*})}{z} dw \quad (9.1.14)$$

Difference and Sum Theorems

$$\mathcal{Z}[\Delta f(x)] = (z - 1)F(z) - zf(0) \quad \text{Differencing} \quad (9.1.15)$$

$$\mathcal{Z}[\Delta^n f(x)] = (z - 1)^n F(z) - z \sum_{k=0}^{n-1} (z - 1)^{n-1-k} f^{(n)}(0) \quad \text{Differencing} \quad (9.1.16)$$

$$\mathcal{Z} \left[\sum_{k=0}^x f(k) \right] = \frac{zF(z)}{z - 1} \quad \text{Summation} \quad (9.1.17)$$

$$\mathcal{Z} \left[\sum_{k=0}^{x-1} f(k) \right] = \frac{F(z)}{z - 1} \quad \text{Summation} \quad (9.1.18)$$

$$\mathcal{Z} \left[\sum_{k=0}^{\infty} f(x - k) \right] = \frac{zF(z)}{z - 1} \quad \text{Summation} \quad (9.1.19)$$

Relation to the Laplace Transform Here, $\mathcal{L}[f(x)]$ denote the one-sided Laplace transform (with respect to x) with the new variable s . We use the notation \mathcal{Z}_s to denote the z -transform with respect to variable s .

$$\mathcal{Z}_s[\mathcal{L}[f(x)]] = \int_0^\infty \frac{f(t)}{1 - e^{-t}z^{-1}} dt \quad (9.1.20)$$

$$s\mathcal{L}[f(x)] = \left(\frac{z - 1}{z} \mathcal{Z}[f(x)] \right)_{z=e^s} \quad (9.1.21)$$

Limit Theorems

$$f(0) = \lim_{z \rightarrow \infty} F(z) \quad \text{Initial Value Theorem} \quad (9.1.22)$$

$$f(n) = \lim_{z \rightarrow \infty} \left[F(z) - \sum_{k=0}^{n-1} f(k)z^{-k} \right] \quad (9.1.23)$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{z \rightarrow 1} (1 - z^{-1})F(z) \quad \text{Final Value Theorem} \quad (9.1.24)$$

Multiplication with Other Functions

$$\mathcal{Z}[a^x f(x)] = F\left(\frac{z}{a}\right) \quad \text{Damping} \quad (9.1.25)$$

$$\mathcal{Z}[\cos(ax)f(x)] = \frac{1}{2} [F(e^{ai}z) + F(e^{-ai}z)] \quad (9.1.26)$$

$$\mathcal{Z}[\sin(ax)f(x)] = \frac{i}{2} [F(e^{ai}z) - F(e^{-ai}z)] \quad (9.1.27)$$

$$\mathcal{Z}[x^k f(x)] = -z \frac{d}{dz} \mathcal{Z}[x^{k-1} f(x)] \quad \text{Differentiation} \quad (9.1.28)$$

$$\mathcal{Z}\left[\frac{f(x)}{x}\right] = \int_z^\infty \frac{F(w)}{w} dw \quad \text{if } f(0) = 0 \quad \text{Integration} \quad (9.1.29)$$

Inverse Formulas For $m \in N$,

$$f(m) = \frac{1}{m!} \left. \frac{d^m}{dz^m} F\left(\frac{1}{z}\right) \right|_{z=0} \quad (9.1.30)$$

9.2. Transform Pairs

Here, we use the following definitions:

9.2.1. Basics

$$\Delta(x) \rightarrow 1 \quad (9.2.1)$$

$$\Delta(x - n) \rightarrow z^{-n} \quad |z| > 0 \quad (9.2.2)$$

$$2 \sum_{k=1}^{\infty} \Delta(x - 2k + 1) \rightarrow \frac{2z}{z^2 - 1} \quad |z| > 1 \quad (9.2.3)$$

$$2 \sum_{k=1}^{\infty} x \Delta(x - 2k + 1) \rightarrow \frac{2z(z^2 + 1)}{(z^2 - 1)^2} \quad |z| > 1 \quad (9.2.4)$$

$$2 \sum_{k=1}^{\infty} \frac{1}{x} \Delta(x - 2k + 1) \rightarrow \ln \frac{z - 1}{z + 1} \quad |z| > 1 \quad (9.2.5)$$

$$(9.2.6)$$

$$1 \rightarrow \frac{z}{z-1} \quad |z| > 1 \quad (9.2.7)$$

$$u(x) \rightarrow \frac{z}{z-1} \quad |z| > 1 \quad (9.2.8)$$

$$u(x-k) \rightarrow \frac{z}{z^k(z-1)} \quad (9.2.9)$$

$$a^x \rightarrow \frac{z}{z-a} \quad |z| > |a| \quad (9.2.10)$$

$$\mathcal{F}(x) \rightarrow \frac{z}{z^2 - z - 1} \quad (9.2.11)$$

$$x \rightarrow \frac{z}{(z-1)^2} \quad (9.2.12)$$

$$x^2 \rightarrow \frac{z(z+1)}{(z-1)^3} \quad |z| > 1 \quad (9.2.13)$$

$$x^3 \rightarrow \frac{z(z^2 + 4z + 1)}{(z-1)^4} \quad |z| > 1 \quad (9.2.14)$$

$$x^4 \rightarrow \frac{z(z^3 + 11z^2 + 11z + 1)}{(z-1)^5} \quad (9.2.15)$$

$$x^5 \rightarrow \frac{z(z^4 + 26z^3 + 66z^2 + 26z + 1)}{(z-1)^6} \quad (9.2.16)$$

$$x^{\langle n \rangle} \rightarrow \frac{n!z}{(z-1)^{n+1}} \quad (9.2.17)$$

$$xa^x \rightarrow \frac{az}{(z-a)^2} \quad |z| > |a| \quad (9.2.18)$$

$$x^2 a^x \rightarrow \frac{az(z+a)}{(z-a)^3} \quad |z| > |a| \quad (9.2.19)$$

$$x^3 a^x \rightarrow \frac{az(z^2 + 4az + a^2)}{(z-a)^4} \quad (9.2.20)$$

$$\frac{1}{x+1} \rightarrow z \ln \left(\frac{z}{z-1} \right) \quad |z| > 1 \quad (9.2.21)$$

$$\frac{a^{x+1}}{x+1} \rightarrow z \ln \left(\frac{z}{z-a} \right) \quad (9.2.22)$$

$$\frac{1-a^{x+1}}{x+1} \rightarrow z \ln \left(\frac{z-a}{z-1} \right) \quad (9.2.23)$$

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^{\langle n \rangle} \rightarrow \sum_{n=0}^{\infty} \frac{a_n z}{(z-1)^{n+1}} \quad (9.2.24)$$

9.2.2. Trigonometric Functions

$$\sin(bx) \rightarrow \frac{z \sin b}{z^2 - 2z \cos b + 1} \quad |z| > 1 \quad (9.2.25)$$

$$\cos(bx) \rightarrow \frac{z(z - \cos b)}{z^2 - 2z \cos b + 1} \quad |z| > 1 \quad (9.2.26)$$

$$a^x \sin(bx) \rightarrow \frac{az \sin b}{z^2 - 2az \cos b + a^2} \quad |z| > |a| \quad (9.2.27)$$

$$a^x \cos(bx) \rightarrow \frac{z(z - a \cos b)}{z^2 - 2az \cos b + a^2} \quad |z| > |a| \quad (9.2.28)$$

$$\frac{\sin b(x+1)}{x+1} \rightarrow z \operatorname{atan} \frac{\sin b}{z - \cos b} \quad (9.2.29)$$

$$\frac{\cos b(x+1)}{x+1} \rightarrow z \ln \frac{z}{\sqrt{z^2 - 2z \cos b + 1}} \quad (9.2.30)$$

9.2.3. Hyperbolic Functions

$$\sinh(bx) \rightarrow \frac{z \sinh b}{z^2 - 2z \cosh b + 1} \quad |z| > \max(|e^b|, |e^{-b}|) \quad (9.2.31)$$

$$\cosh(bx) \rightarrow \frac{z(z - \cosh b)}{z^2 - 2z \cosh b + 1} \quad |z| > \max(|e^b|, |e^{-b}|) \quad (9.2.32)$$

$$a^x \sinh(bx) \rightarrow \frac{az \sinh b}{z^2 - 2az \cosh b + a^2} \quad |z| > \max(|ae^b|, |ae^{-b}|) \quad (9.2.33)$$

$$a^x \cosh(bx) \rightarrow \frac{z(z - a \cosh b)}{1 - 2az \cosh b + a^2} \quad |z| > \max(|ae^b|, |ae^{-b}|) \quad (9.2.34)$$

9.2.4. Discrete Trigonometric Functions

$$\text{sind}(bx) \rightarrow \frac{\text{sind}(b)z^2}{z^2 - 2 \cosd(b)z + 2^b} \quad (9.2.35)$$

$$\cosd(bx) \rightarrow \frac{z(\cosd(b)z - 2^b)}{z^2 - 2 \cosd(b)z + 2^b} \quad (9.2.36)$$

$$a^x \text{sind}(bx) \rightarrow \frac{\text{sind}(b)z^2}{z^2 - 2a \cosd(b)z + a^2 2^b} \quad (9.2.37)$$

$$a^x \cosd(bx) \rightarrow \frac{z(\cosd(b)z - a2^b)}{z^2 - 2a \cosd(b)z + a^2 2^b} \quad (9.2.38)$$

9.2.5. Gamma and Related Functions

$$\Gamma(x) \rightarrow^? e^{-z} C \quad (9.2.39)$$

$$\frac{a^x}{\Gamma(x+1)} \rightarrow e^{a/z} \quad |z| > 0 \quad (9.2.40)$$

$$\frac{(-1)^x}{\Gamma(2x+2)} \rightarrow \sqrt{z} \sin \frac{1}{\sqrt{z}} \quad (9.2.41)$$

$$\frac{(-1)^x}{\Gamma(2x+1)} \rightarrow \sqrt{z} \cos \frac{1}{\sqrt{z}} \quad (9.2.42)$$

$$\frac{(-a)^x}{\Gamma(2x+2)} \rightarrow \sqrt{\frac{z}{a}} \sin \sqrt{\frac{a}{z}} \quad (9.2.43)$$

$$\frac{(-a)^x}{\Gamma(2x+1)} \rightarrow \sqrt{\frac{z}{a}} \cos \sqrt{\frac{a}{z}} \quad (9.2.44)$$

$$\frac{a^{2x+1}}{\Gamma(2x+2)} \rightarrow \sinh \left(\frac{a}{z} \right) \quad (9.2.45)$$

$$\frac{a^{2x}}{\Gamma(2x+1)} \rightarrow \cosh \left(\frac{a}{z} \right) \quad (9.2.46)$$

$$\frac{(\ln a)^x}{\Gamma(x+1)} \rightarrow a^{1/z} \quad (9.2.47)$$

$$\binom{y}{x} a^{y-x} b^x \rightarrow \frac{(az+b)^y}{z^y} \quad (9.2.48)$$

$$\binom{x+y}{y} b^x \rightarrow \frac{z^{y+1}}{(z-b)^{y+1}} \quad (9.2.49)$$

$$\Gamma^{(-1)}(x) \rightarrow^? \frac{e^{-z} C}{z-1} \quad (9.2.50)$$

$$x\Gamma^{(-1)}(x) \rightarrow^? \frac{e^{-z} z^2 C}{(z-1)^2} \quad (9.2.51)$$

$$\psi(x+1) - \psi(1) \rightarrow \frac{z}{z-1} \ln \left(\frac{z}{z-1} \right) \quad (9.2.52)$$

$$B(x, a) \rightarrow^? (z-1)^{a-1} C \quad (9.2.53)$$

$$\gamma(x, a) \rightarrow^? \text{Ei}(a-z)e^z + Ce^z \quad (9.2.54)$$

$$\Gamma(x, a) \rightarrow^? -\text{Ei}(a-z)e^z + Ce^z \quad (9.2.55)$$

9.2.6. Exponential Sums

$$\text{Bs}(b(x+1)) \rightarrow \frac{z^2}{b(z-1)} \ln \left(\frac{z}{z-2^b} \right) \quad (9.2.56)$$

$$\text{Ss}(b(x+1)) \rightarrow \frac{z^2}{b(z-1)} \tan \left(\frac{\sin b}{z-\cos b} \right) \quad (9.2.57)$$

$$\text{Cs}(b(x+1)) \rightarrow \frac{z^2}{b(z-1)} \ln \left(\frac{z}{\sqrt{z^2 - 2\cos(b)z + 1}} \right) \quad (9.2.58)$$

$$\text{Sds}(b(x+1)) \rightarrow \frac{z^2}{b(z-1)} \tan \left(\frac{\sinh b}{z-\cosh b} \right) \quad (9.2.59)$$

$$\text{Cds}(b(x+1)) \rightarrow \frac{z^2}{b(z-1)} \ln \left(\frac{z}{\sqrt{z^2 - 2\cosh(b)z + 2^b}} \right) \quad (9.2.60)$$

9.2.7. Orthogonal Polynomials

$$a^x P_x(t) \rightarrow \frac{z}{\sqrt{z^2 - 2atz + a^2}} \quad (9.2.61)$$

$$a^x P_x^m(t) \rightarrow \frac{(2m)! z^{m+1} (1-t^2)^{m/2} a^m}{2^m m! (z^2 - 2atz + a^2)^{m+\frac{1}{2}}} \quad (9.2.62)$$

$$\frac{P_x(t)}{x!} \rightarrow \exp \left(\frac{t}{z} \right) J_0 \left(\frac{\sqrt{1-x^2}}{z} \right) \quad (9.2.63)$$

$$\frac{P_x^m(t)}{(x+m)!} \rightarrow (-1)^m \exp \left(\frac{t}{z} \right) J_m \left(\frac{\sqrt{1-x^2}}{z} \right) \quad (9.2.64)$$

$$a^x T_x(t) \rightarrow \frac{z(z-at)}{z^2 - 2atz + a^2} \quad (9.2.65)$$

$$\frac{L_x(t)}{x!} \rightarrow \frac{ze^{-t/(z-1)}}{z-1} \quad (9.2.66)$$

$$\frac{H_x(t)}{x!} \rightarrow e^{t/z - \frac{1}{2}z^2} \quad (9.2.67)$$

$$\frac{L_x^m(t)}{x!} \rightarrow \frac{(-1)^m z}{(z-1)^{m+1}} \exp \left(\frac{-t}{z-1} \right) \quad (9.2.68)$$

$$(9.2.69)$$

9.2.8. Sums of Orthogonal Polynomials

$$\sum_{k=0}^{x-1} \frac{1}{\Gamma(k+1)} \rightarrow \frac{e^{1/z}}{z-1} \quad (9.2.70)$$

$$\sum_{k=0}^x \frac{a^k b^{x-k}}{k!} \rightarrow \frac{e^{a/z} z}{z-b} \quad b^2 < 1 \quad (9.2.71)$$

$$\sum_{k=0}^x a^k b^{x-k} J_k(t) \rightarrow \frac{z}{z-b} \exp\left(\frac{t(a^2 + z^2)}{2az}\right) \quad b^2 < 1; d \in \mathbb{R} \quad (9.2.72)$$

$$\sum_{k=0}^x \frac{a^k b^{x-k}}{k!} H_k(t) \rightarrow \frac{z}{z-b} \exp\left(-\frac{dz(2tz+a)}{2z^2}\right) \quad b^2 < 1 \quad (9.2.73)$$

$$\sum_{k=0}^x \frac{a^k b^{x-k}}{k!} \frac{d}{dt} L_k(t) \rightarrow \frac{(-a)^m z^2}{(z-a)^{m+1}(z-b)} \exp\left(\frac{at}{1-z}\right) \quad a^2 < 1; b^2 < 1 \quad (9.2.74)$$

$$\sum_{k=0}^x \frac{b^{x-k} [(-a)^k - (-c)^k]}{k} \rightarrow \frac{z}{z-b} \ln\left(\frac{z+c}{z+a}\right) \quad a^2 < 1; b^2 < 1; c^2 < 1 \quad (9.2.75)$$

$$\sum_{k=0}^x a^k b^{x-k} \frac{d}{dt} P_k(t) \rightarrow \frac{(2m)! a^m}{2^m m!} \frac{z^{m+2}}{(z-b)(z^2 - 2atz + a^2)^{m+\frac{1}{2}}} \quad a^2 < 1; b^2 < 1 \quad (9.2.76)$$

$$\sum_{k=0}^x a^k b^{x-k} T_k(t) \rightarrow \frac{z^2(z-at)}{(z-bz)(z^2 - 2atz + a)} \quad a^2 < 1; b^2 < 1 \quad (9.2.77)$$

10. Binomial Transforms

10.1. Definition

The binomial transform $F(k) = \mathcal{B}[f(x)]$ of a function $f(x)$ is defined by:

$$F(k) = \sum_{x=0}^k (-1)^{k-x} \binom{k}{x} f(x) \quad (10.1.1)$$

The inverse transform $f(x) = \mathcal{B}^{-1}[F(x)]$ of a function $f(x)$ is defined by:

$$f(x) = \sum_{k=0}^x \binom{x}{k} F(k) \quad (10.1.2)$$

10.2. Properties

Here, E is the shift operator: $E_x f(x) = f(x - 1)$.

$$\mathcal{B}[f(x)] = f^{\langle k \rangle}(0) \quad (10.2.1)$$

$$\mathcal{B}[af(x) + bg(x)] = a\mathcal{B}[f(x)] + b\mathcal{B}[g(x)] \quad (10.2.2)$$

$$\mathcal{B}^{-1}[E_k f(k)] = f(-1) + x E_x \mathcal{B}^{-1} \left[\frac{f(k)}{k+1} \right] \quad (10.2.3)$$

$$\mathcal{B}^{-1}[k E_k f(k)] = x E_x \mathcal{B}^{-1}[f(k)] \quad (10.2.4)$$

$$\mathcal{B}^{-1} \left[\sum_{m=0}^{\infty} a_m k^{\langle m \rangle} \right] = 2^x \sum_{m=0}^{\infty} a_m 2^{-m} x^{\langle m \rangle} \quad (10.2.5)$$

$$f(x) = \sum_{k=0}^{\infty} F(k) \frac{x^{\langle k \rangle}}{k!} \quad (10.2.6)$$

$$F(k) = \Delta^k f(x)|_{x=0} \quad (10.2.7)$$

10.3. Pairs

10.3.1. Basics

$$1 \rightarrow \Delta(k) \quad (10.3.1)$$

$$x \rightarrow \Delta(k - 1) \quad (10.3.2)$$

$$x^{\langle n \rangle} \rightarrow n! \Delta(k - n) \quad (10.3.3)$$

$$x! \rightarrow !k \quad (10.3.4)$$

$$2^x \rightarrow 1 \quad (10.3.5)$$

$$a^x \rightarrow (a - 1)^k \quad (10.3.6)$$

$$\frac{1}{2^x} \rightarrow (-1)^k \frac{1}{2^k} \quad (10.3.7)$$

10.4. Pairs (Inverse)

10.4.1. Basics

$$1 \rightarrow 2^x \quad (10.4.1)$$

$$k \rightarrow x2^{x-1} \quad (10.4.2)$$

$$k^{\langle n \rangle} \rightarrow x^{\langle n \rangle} 2^{x-n} \quad n \geq 0 \quad (10.4.3)$$

$$k^{\langle -1 \rangle} \rightarrow \frac{1}{x+1} (2^{x+1} - 1) \quad (10.4.4)$$

$$k^{\langle -2 \rangle} \rightarrow x^{\langle -2 \rangle} (2^{x+2} - x - 3) \quad (10.4.5)$$

$$k^{\langle -3 \rangle} \rightarrow x^{\langle -3 \rangle} \left(2^{x+3} - \frac{x^2 + 9x + 14}{2} \right) \quad (10.4.6)$$

$$k^{\langle -n \rangle} \rightarrow x^{\langle -2 \rangle} \left(2^{x+n} - \sum_{k=0}^{n-1} \binom{x+n}{k} \right) \quad (10.4.7)$$

$$a^k \rightarrow (a+1)^x \quad (10.4.8)$$

10.4.2. Discrete Trigonometric Functions

$$\sin k \rightarrow 2^x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} 2^{-2m-1} x^{\langle 2m+1 \rangle} \quad (10.4.9)$$

$$\cos k \rightarrow 2^x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} 2^{-2m} x^{\langle 2m \rangle} \quad (10.4.10)$$

$$(10.4.11)$$

10.4.3. Gamma and Related Functions

$$\psi(k+1) \rightarrow \gamma 2^x + 2^x \sum_{m=1}^{\infty} (-1)^m (m-1)! 2^{-m} x^{\langle m \rangle} \quad (10.4.12)$$

$$\Gamma(k+1) \rightarrow \sum_{k=0}^x x^{\langle k \rangle} \quad (10.4.13)$$

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