Reference for Discrete Calculus and Functional Equations

Version 1.2

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Preface

Version 1

This document started of as notes for my own work. After getting too frustrated by not being able to find auxiliary formulas in my (many) notebooks and having to derive them repeatedly, I decided to compile a neat reference for myself. This is very much a work in progress. This version contains some gaping holes and is admittedly sloppy in many respects. In particular, watch out for these issues:

- The discrete Taylor series might not converge everywhere. For example, the expansion of the Gamma function only converges to the Gamma function for integer x. In addition, the radius of convergence has been omitted everywhere.
- In many cases the constant difference has been taken as $h = 1$. This has been indicated in some places, but not in all.
- In the section on standard functions of difference operators, I have not specified all the conditions for theorems where inverse functions are involved (these work "straight out of the box" only for bijections).
- The z-transform table contains some formulas that have been formally derived, without checking whether the series (of the definition) actually converges. This has been indicated by little question marks next to the transformation arrows. The range of convergence has been omitted everywhere.

Furthermore, I used some non-standard notations, and there are many inconsistencies in style or form that make this reference hard to read.

I intend to address these in coming versions; in the mean time, even with all its flaws, some might still find this reference useful.

Version 1.1

- Added Exponential Sums to differences and sums.
- Additions to the z-transform table.
- Added Binomial Transform pairs.

Version 1.2

- Expanded the section on the discrete power functions.
- Expanded the section that explain the sue of constants in the table.
- Added forms involving the following expressions to the sum $(x + h)$ tables :
  - $ax + b$
  - $x^2 + a^2$
- Updated all the graphs.
- Reorganized slightly, and fixed some typos.
- Added a few examples, explanations, and additional notations in the sum $(x + h)$ tables.

1. Introduction

1.1. Introduction

This document contains tables and formulas useful for working with functional equations, especially difference equations, and to a lessor extent, quotient equations.

2. Functions

2.1. Discrete Power Function

The discrete power function have properties in discrete calculus analogous to the standard power function in standard calculus. For example, see (3.19). The function is also sometimes called the *falling factorial*.

2.1.1. Definitions

$$x^{\langle a\rangle} = \frac{\Gamma(x+1)}{\Gamma(x+1-a)} \qquad x \neq -1,-2,\ldots \quad x-a \neq -1,-2,-3,\ldots \tag{2.1}$$

$$x^{\langle a\rangle_h} = h^a \frac{\Gamma\left(\frac{x}{h}+1\right)}{\Gamma\left(\frac{x}{h}+1-a\right)} \qquad \frac{x}{h} \neq -1,-2,-3,\ldots \quad \frac{x}{h}-a \neq -1,-2,-3,\ldots \tag{2.2}$$

In cases where the Gamma function is infinite, the limit is taken if it exists:

$$x^{\langle a\rangle} = \lim_{t\to x} \frac{\Gamma(t+1)}{\Gamma(t+1-a)} \tag{2.3}$$

$$x^{\langle a\rangle_h} = \lim_{t\to x} h^a \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-a\right)} \tag{2.4}$$

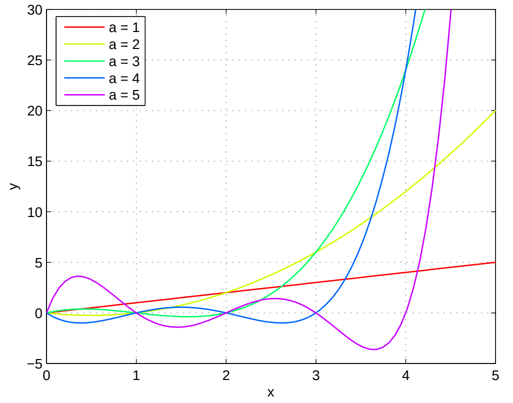
Note that $x^{\langle a\rangle}$ is just the special case of $x^{\langle a\rangle_h}$ when $h=1$.

Alternative notation The following notation is commonly used instead of the $x^{\langle n\rangle}$ presented above: $x^{\underline{a}}$, $x^{(a)}$, $(x)_a$.

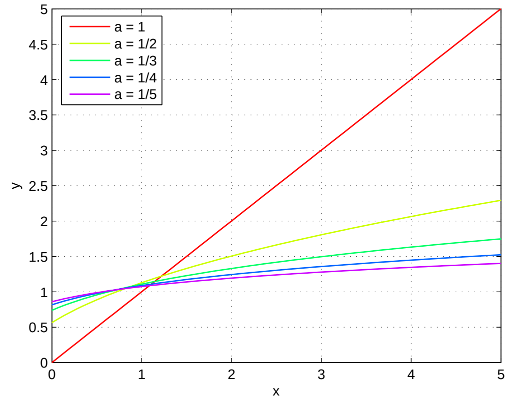
Examples

$$6^{\langle 3\rangle} = \frac{\Gamma(6+1)}{\Gamma(6+1-3)} = \frac{720}{6} = 120 \qquad 6^{\langle 3\rangle_2} = 2^3\frac{\Gamma(\frac{6}{2}+1)}{\Gamma(\frac{6}{2}+1-3)} = 8\cdot\frac{6}{1} = 48$$

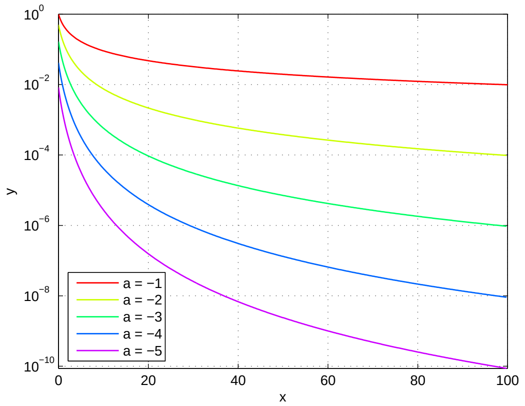
$$\left(-\frac{1}{2}\right)^{\langle -1\rangle} = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\sqrt{\pi}}{\frac{3}{2}\sqrt{\pi}} = \frac{2}{3} \qquad \left(-\frac{1}{2}\right)^{\langle -1\rangle_{\frac{1}{2}}} = 2^{-1}\frac{\Gamma(2)}{\Gamma(5)} = \frac{1}{2}\cdot\frac{1}{24} = \frac{1}{48}$$



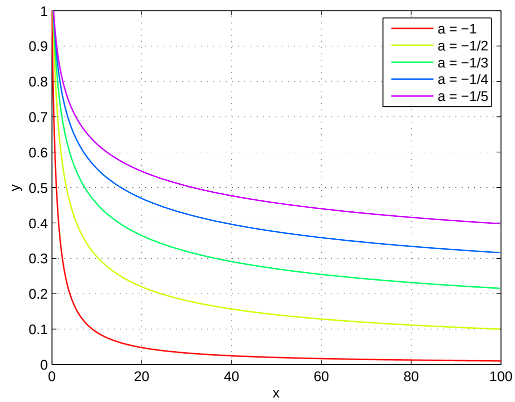
(a) Positive integers



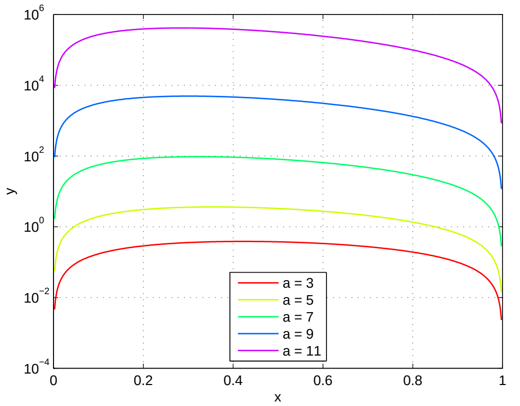
(b) Positive integer reciprocals



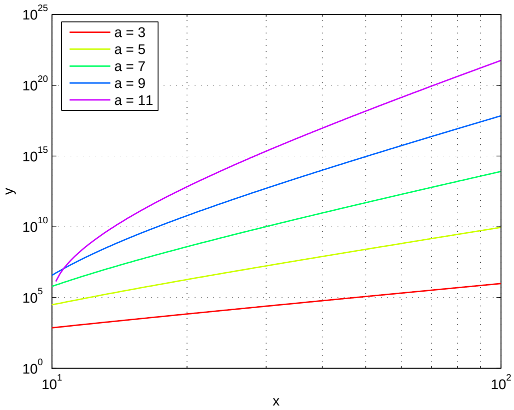
(c) Negative integers



(d) Negative integer reciprocals



(e) Odd integers



(f) Positive integers

Figure 2.1.: Discrete Power Function. $y = x^{\langle a \rangle}$

2.1.2. Properties

For integer n:

$$x^{\langle n\rangle} = x(x-1)(x-2)...(x-n+1) \qquad n>0 \tag{2.5}$$

$$x^{\langle -n\rangle} = \frac{1}{(x+1)(x+2)...(x+n)} \qquad n>0 \tag{2.6}$$

$$x^{\langle 0\rangle} = 1 \tag{2.7}$$

$$x^{\langle n\rangle_h} = x(x-h)(x-2h)...(x-(n-1)h) \tag{2.8}$$

$$x^{\langle -n\rangle_h} = \frac{1}{(x+h)(x+h)...(x+nh)} \tag{2.9}$$

$$x^{\langle 0\rangle_h} = 1 \tag{2.10}$$

$$\prod_{k=0}^{n-1}\left(x-\frac{k}{n}\right)^{\left\langle\frac{1}{n}\right\rangle} = x \tag{2.11}$$

$$x^{\langle n\rangle} = (-1)^{n+1}\Gamma(x+1)\Gamma(n-x)\sin\pi x \qquad x\notin\mathbb{Z} \tag{2.12}$$

$$(-x)^{\langle n\rangle} = (-1)^n(x+n-1)^{\langle n\rangle} \tag{2.13}$$

$$?(-x)^{\langle n+\frac{1}{2}\rangle} = (-1)^n\left(x+n-\frac{1}{2}\right)^{\langle n+\frac{1}{2}\rangle}\tan\pi x = 0 \tag{2.14}$$

$$1^{\langle n\rangle} = \begin{cases} 0 & n>1 \\ 1 & n=1 \\ \frac{1}{(n+1)!} & n<1 \end{cases} \tag{2.15}$$

$$0^{\langle n\rangle} = \begin{cases} 0 & n>0 \\ 1 & n=0 \\ \frac{1}{n!} & n<0 \end{cases} \tag{2.16}$$

Note that (2.7) and (2.10) also holds when $x=0$, that is, $0^{\langle 0\rangle}=1$.

For real a:

$$x^{\langle a\rangle}(x-a)^{\langle b\rangle} = x^{\langle a+b\rangle} \tag{2.17}$$

$$x^{\langle -a\rangle} = \frac{1}{(x+a)^{\langle a\rangle}} \tag{2.18}$$

$$x^{\langle a\rangle_h}(x-ah)^{\langle b\rangle} = x^{\langle a+b\rangle} \tag{2.19}$$

$$x^{\langle -a\rangle_h} = \frac{1}{(x+ah)^{\langle a\rangle_h}} \tag{2.20}$$

$$x^{\langle a\rangle} = -\Gamma(x+1)\Gamma(a-x)\sin(a-x)\pi \qquad a\notin\mathbb{Z} \tag{2.21}$$

$$(-x)^{\langle a\rangle} = (x+a-1)^{\langle a\rangle}(\cos a\pi + \sin a\pi\tan\pi x) \qquad a\notin\mathbb{Z} \tag{2.22}$$

$$1^{\langle a\rangle} = \Gamma(a-1)\sin\pi a \qquad a\neq -1,-2,-3,\ldots \tag{2.23}$$

$$0^{\langle a\rangle} = -\Gamma(a)\sin\pi a \qquad a\neq 0,-1,-2,\ldots \tag{2.24}$$

$$x^{\langle x\rangle} = \Gamma(x+1) \tag{2.25}$$

$$\tag{2.26}$$

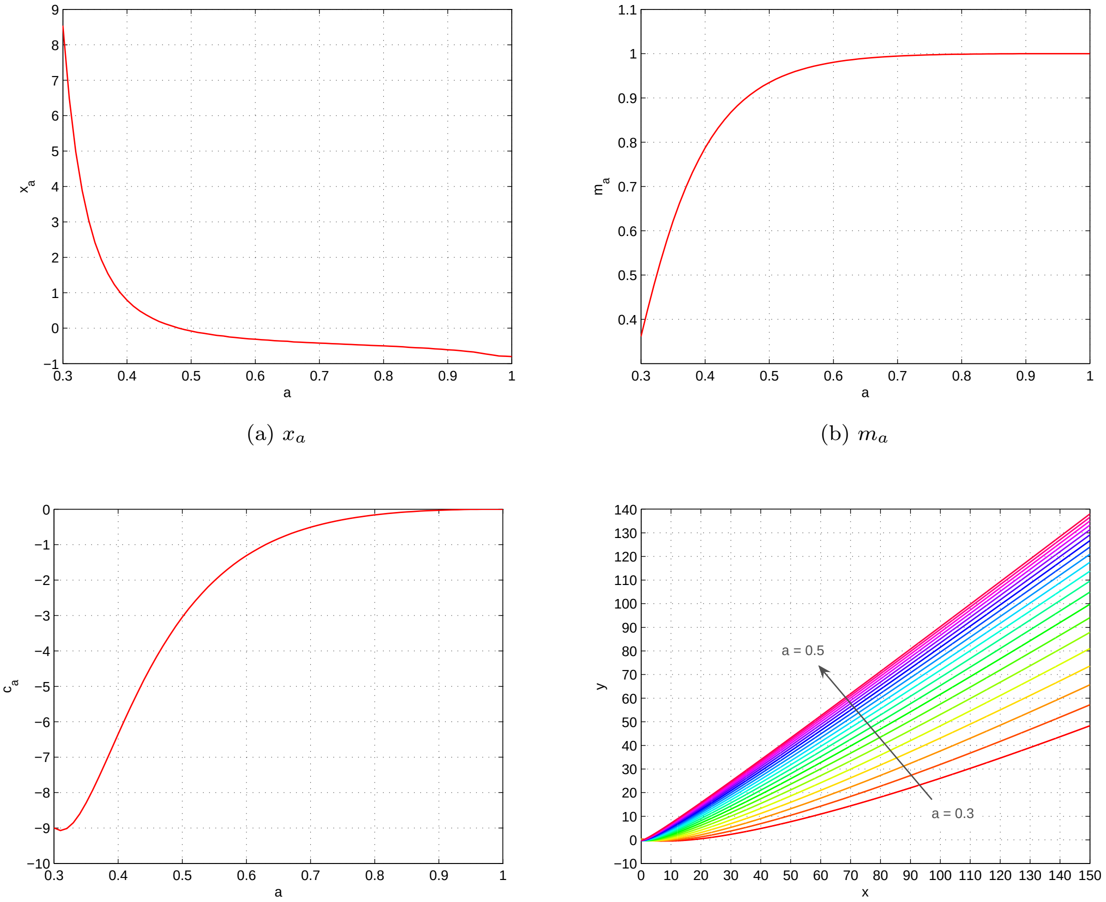
The following approximation mimics the formula $(x^a)^{1/a}=x$ that we have for standard powers:

$$(x^{\langle a\rangle})^{\langle 1/a\rangle} \approx m_a x + c_a \text{ for } x > x_a \tag{2.27}$$

The following results have been obtained numerically for $1\le a\le 2.5$ and $x_a\le x\le 5$. It is possible that it holds for larger x and larger a.

$$x_a \approx 1.0519a - 1.5814 \tag{2.28}$$
$$m_a \approx 0.062898a^2 - 0.22811a + 1.1753 \tag{2.29}$$
$$c_a \approx -0.2793a^2 + 0.44094a - 0.152 \tag{2.30}$$

The curves in Figure 2.2 gives values for x_a, m_a and c_a when $0.3 \leq a \leq 1$.



(a) x_a (b) m_a (c) c_a (d) $y = (x^{\langle a \rangle})^{\langle 1/a \rangle}$ for $a = 0.3 : 0.01 : 0.5$

Figure 2.2.: Curves used in the inverse power approximation.

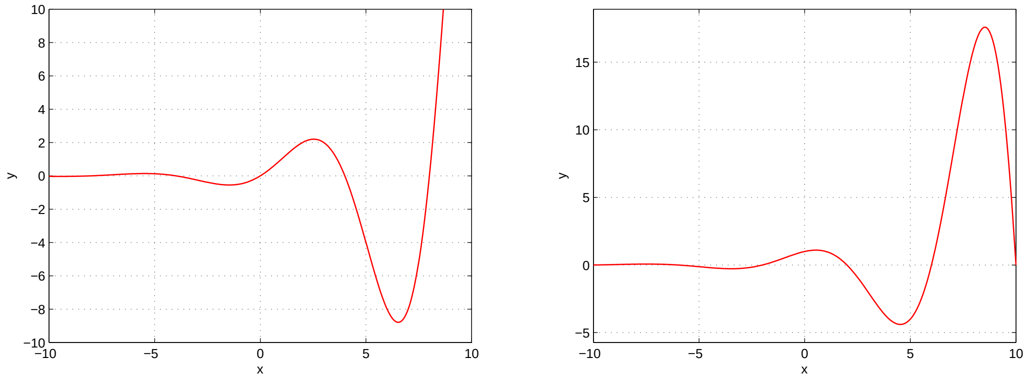
$$\tag{2.31}$$

2.2. Discrete Trigonometric Functions

Discrete trigonometric functions are the discrete counterparts of standard trigonometric functions. For example, see (3.50) and (3.49).

2.2.1. Definition

$$\operatorname{sind} x = 2^{x/2} \sin\left(\frac{\pi x}{4}\right) \tag{2.32}$$
$$\operatorname{cosd} x = 2^{x/2} \cos\left(\frac{\pi x}{4}\right) \tag{2.33}$$
$$\operatorname{cisd} x = 2^{x/2} \operatorname{cis}\left(\frac{\pi x}{4}\right) \tag{2.34}$$



(a) $\operatorname{sind} x$ (b) $\operatorname{cosd} x$

Figure 2.3.: Discrete Trigonometric Functions

2.2.2. Properties

$$\frac{\operatorname{sind} x}{\operatorname{cosd} x} = \tan\frac{\pi x}{4} \tag{2.35}$$
$$\frac{\operatorname{cosd} x}{\operatorname{sind} x} = \cot\frac{\pi x}{4} \tag{2.36}$$
$$\operatorname{cisd} x = (2i)^{\frac{x}{2}} \tag{2.37}$$

$$\operatorname{sind}(x+y) = \operatorname{sind}(x)\operatorname{cosd}(y) + \operatorname{cosd}(x)\operatorname{sind}(y) \tag{2.38}$$
$$\operatorname{sind}(x-y) = 2^y \operatorname{sind}(x)\operatorname{cosd}(y) - 2^y \operatorname{cosd}(x)\operatorname{sind}(y) \tag{2.39}$$
$$\operatorname{cosd}(x+y) = \operatorname{cosd}(x)\operatorname{cosd}(y) - \operatorname{sind}(x)\operatorname{sind}(y) \tag{2.40}$$
$$\operatorname{cosd}(x-y) = 2^y \operatorname{cosd}(x)\operatorname{cosd}(y) + 2^y \operatorname{sind}(x)\operatorname{sind}(y) \tag{2.41}$$

$$\operatorname{sind}^2(x) + \operatorname{cosd}^2(x) = 2^x \tag{2.42}$$

$$\operatorname{sind}(2x) = 2\operatorname{sind}(x)\operatorname{cosd}(x) \tag{2.43}$$
$$\operatorname{cosd}(2x) = \operatorname{cosd}^2(x) - \operatorname{sind}^2(x) \tag{2.44}$$
$$= 2\operatorname{cosd}^2(x) - 2^x \tag{2.45}$$
$$= 2^x - 2\operatorname{sind}^2(x) \tag{2.46}$$

2.3. Fibonacci Function

2.3.1. Definition

$$\mathcal{F}(x) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x\right] \tag{2.47}$$

2.3.2. Properties of the Fibonacci Function

$$\mathcal{F}(x) = \mathcal{F}(x-1) + \mathcal{F}(x-2) \tag{2.48}$$

$$\mathcal{F}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}(x-k-m) \tag{2.49}$$

2.4. G

G is the sum of the Gamma function.

$$G(x+1) = \sum_{k=0}^{\infty} !k \frac{x^{\langle k \rangle}}{k!} \qquad !n = 1, 0, 1, 2, 9, 44, 265, 1854 \tag{2.50}$$

2.5. The Hyper Power Function

A function that satisfies

$$H_a(x+1) = a^{H_a(x)} \tag{2.51}$$

and

$$H_a(0) = 1. \tag{2.52}$$

An inverse hyperpower function satisfies

$$\operatorname{slog}_a(a^x) = \operatorname{slog}_a(x) + 1. \tag{2.53}$$

Part I.

Discrete Calculus Tables

Constants

Constants can generally be replaced by any function with appropriate properties. For instance, the table gives

$$\sum a^x = \frac{a^x}{a-1} + C.$$

Since, for any period function $\Theta(x)$ with period 1 we have $\Delta\Theta(x) = \Theta(x+1) - \Theta(x) = \Theta(x) - \Theta(x) = 0$, we can actually replace the constant C with any function with period 1, that is

$$\sum a^x = \frac{a^x}{a-1} + \Theta(x).$$

We can also replace the a with an arbitrary periodic function (with period 1), so that we have:

$$\sum (\Theta_1(x))^x = \frac{(\Theta_1(x))^x}{\Theta_1(x) - 1} + \Theta_2(x).$$

Similarly, the table gives:

$$\mathrm{P}^{-1}\, x = \frac{x}{h-1} + C.$$

Here we can replace C with any function that satisfies $C(hx) = C(x)$. If $\Theta(x)$ is a periodic function with period 1, then $\Theta(\log_h x)$, satisfies this condition, so:

$$\mathrm{P}^{-1}\, x = \frac{x}{h-1} + \Theta(\log_h x).$$

In the general case, the constant for operator Δ_g can be replaced by $\Theta(I_\Delta g(x))$, where Θ is any periodic function with period 1 (see formula 7.9).

3. Sum $(x+h)$

3.1. Definition and Properties

3.1.1. Definition

$$\Delta f(x) = f(x+h) - f(x) \tag{3.1}$$
$$\Sigma = \Delta^{-1} \tag{3.2}$$

Notation When x or x and h are not clear from the context, they can be specified with the symbols: $\Delta_{x,h} f(x)$ and $\Sigma_x f(x)$. The notation can also mimic that of standard calculus:

$$\frac{\Delta}{\Delta x} f(x) \qquad \sum f(x)\Delta x \tag{3.3}$$

The n difference of a function f can be denoted $f^{\langle n\rangle}(x) = \Delta^n f(x)$, and is the function obtained when applying the difference operator n times. The notation $\Delta f(x)|_{x=a}$ is used to denote $\Delta f(x)$ evaluated at $x = a$.

Examples

$$\Delta x^2 = (x+1)^2 - x^2 = 2x+1 \qquad \Sigma(2x+1) = x^2 + C$$
$$\Delta\Gamma(x) = \Gamma(x+1) - \Gamma(x) = x\Gamma(x) - \Gamma(x) = (x-1)\Gamma(x) \qquad \Sigma(x-1)\Gamma(x) = \Gamma(x) + C$$

3.1.2. Properties of Operators

$$\Delta(af(x) + bg(x)) = a\Delta f(x) + b\Delta g(x) \quad \text{Linearity} \tag{3.4}$$
$$\Delta(f(x)g(x)) = f(x+1)\Delta g(x) + g(x)\Delta f(x) \quad \text{Product Rule} \tag{3.5}$$
$$\Delta \sum f(x) = f(x) \quad \text{Linearity} \tag{3.6}$$
$$\sum_a^{b-1} f(x) = \sum f(x)\Big|_a^b \quad \text{Fundamental Theorem} \tag{3.7}$$
$$\sum f(x)g(x) = f(x)\sum g(x) - \sum \Delta f(x) \sum g(x+1) \quad \text{Summation by Parts} \tag{3.8}$$
$$f^{\langle n\rangle}(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f(x+k) \tag{3.9}$$

Here $\Theta(x)$ is a periodic function with period h, that is, $\Theta(x+h) = \Theta(x)$.

$$\Delta\Theta(x) = 0 \tag{3.10}$$
$$\Delta\Theta(x)f(x) = \Theta(x)\Delta f(x) \tag{3.11}$$
$$\sum \Theta(x)f(x) = \Theta(x)\sum f(x) \tag{3.12}$$

3.1.3. Analogues of Elementary Functions

Below $\Theta(x)$ is *any* periodic function with period h.

Elementary Function	Analog
c	$\Theta(x)$
x^a	$x^{\langle a\rangle}$
$\sqrt{x}$	$x^{\langle\frac{1}{2}\rangle}$
e^x	2^x
$\sin x$	$\operatorname{sind} x$
$\cos x$	$\operatorname{cosd} x$
$\ln x$	$\psi_0(x)$

3.2. Differences

3.2.1. Powers

$$\Delta c = 0 \tag{3.13}$$

$$\Delta ax = ah \tag{3.14}$$

$$\Delta x^2 = 2hx + h^2 \tag{3.15}$$

$$\Delta x^3 = 3hx^2 + 3h^2x + h^3 \tag{3.16}$$

$$\Delta x^4 = 4hx^3 + 6h^2x^2 + 4h^3x + h^4 \tag{3.17}$$

$$\Delta x^n = \sum_{k=0}^{n-1} \binom{n}{k} x^k h^{n-k} \tag{3.18}$$

$$\Delta x^{\langle a\rangle_h} = ax^{\langle a-1\rangle_h} \tag{3.19}$$

3.2.2. Binomial Coefficients

$$\Delta \binom{x}{n}_h = h\binom{x}{n-1}_h \tag{3.20}$$

$$\Delta \binom{-x}{n} = \binom{-x-1}{n-1} \qquad h=1 \tag{3.21}$$

3.2.3. Exponential Functions

$$\Delta a^x = a^x(a^h-1) \tag{3.22}$$

$$\Delta \log_a x = \log_a\left(1+\frac{h}{x}\right) \tag{3.23}$$

$$\Delta a^{\langle x\rangle} = a^{\langle x\rangle}(x+1-a) \qquad h=1 \tag{3.24}$$

$$\Delta \mathcal{F}(x) = \mathcal{F}(x-1) \qquad h=1 \tag{3.25}$$

3.2.4. Trigonometric Functions

$$\Delta \sin ax = 2\sin\frac{ah}{2}\cos\left(ax+\frac{ah}{2}\right) \tag{3.26}$$

$$\Delta \cos ax = -2\sin\frac{ah}{2}\sin\left(ax+\frac{ah}{2}\right) \tag{3.27}$$

$$\Delta \tan ax = \frac{\sin ah}{\cos(ax+ah)\cos ax} \tag{3.28}$$

$$\Delta \cot ax = \frac{\sin ah}{\sin(ax+ah)\sin ax} \tag{3.29}$$

$$\Delta \sec x = \frac{\sin(ax+\frac{ah}{2})\sin(\frac{ah}{2})}{\cos(ax+ah)\cos ax} \tag{3.30}$$

$$\Delta \csc x = -\frac{\cos(ax+\frac{ah}{2})\sin(\frac{ah}{2})}{\sin(ax+ah)\sin ax} \tag{3.31}$$

3.2.5. Inverse Trigonometric Functions

$$\Delta \operatorname{asin} ax = \operatorname{asin}\left(a(x+h)\sqrt{1-a^2x^2} - ax\sqrt{1-a^2(x+h)^2}\right) \tag{3.32}$$

$$\Delta \operatorname{acos} ax = \operatorname{acos}\left(a^2x(x+h) - \sqrt{(1-a^2(x+h)^2)(1-a^2x^2)}\right) \tag{3.33}$$

$$\Delta \operatorname{atan} ax = \operatorname{atan}\left(\frac{ah}{a^2x^2+a^2xh+1}\right) \tag{3.34}$$

$$\Delta \operatorname{acot} ax = -\operatorname{acot}\left(\frac{a^2x^2+a^2xh+1}{ah}\right) \tag{3.35}$$

$$\Delta \operatorname{asec} ax = -\operatorname{asec}\left(\frac{a^2x(x+h)}{1-\sqrt{(a^2x^2-1)(a^2(x+h)^2-1)}}\right) \tag{3.36}$$

$$\Delta \operatorname{acsc} ax = \operatorname{asec}\left(\frac{a^2x(x+h)\sqrt{(1-a^2x^2)(1-a^2(x+h)^2)}}{ax\sqrt{1-a^2(x+h)^2}+a(x+h)\sqrt{1-a^2x^2}}\right) \tag{3.37}$$

3.2.6. Hyperbolic Functions

$$\Delta \sinh x = 2\cosh\left(x+\frac{h}{2}\right)\sinh\left(\frac{h}{2}\right) \tag{3.38}$$

$$\Delta \cosh x = 2\sinh\left(x+\frac{h}{2}\right)\sinh\left(\frac{h}{2}\right) \tag{3.39}$$

$$\Delta \tanh x = \frac{\sinh h}{\cosh(x+h)\cosh(x)} \tag{3.40}$$

$$\Delta \coth x = -\frac{\sinh h}{\sinh(x+h)\sinh(x)} \tag{3.41}$$

$$\Delta \operatorname{sech} x = -\frac{2\sinh(x+\frac{h}{2})\sinh(\frac{h}{2})}{\cosh(x+h)\cosh(x)} \tag{3.42}$$

$$\Delta \operatorname{csch} x = -\frac{2\cosh(x+\frac{h}{2})\sinh(\frac{h}{2})}{\sinh(x+h)\sinh(x)} \tag{3.43}$$

3.2.7. Exponential Sums

$$\Delta\,\mathrm{Bs}(x) = \frac{2^x}{x} \tag{3.44}$$
$$\Delta\,\mathrm{Ss}(x) = \frac{\operatorname{sind} x}{x} \tag{3.45}$$
$$\Delta\,\mathrm{Cs}(x) = \frac{\operatorname{cosd} x}{x} \tag{3.46}$$
$$\Delta\,\mathrm{Shs}(x) = \frac{\operatorname{sinhd} x}{x} \tag{3.47}$$
$$\Delta\,\mathrm{Chs}(x) = \frac{\operatorname{coshd} x}{x} \tag{3.48}$$

3.2.8. Discrete Additive Trigonometric Functions

$$\Delta \operatorname{sind} x = \operatorname{cosd} x \tag{3.49}$$
$$\Delta \operatorname{cosd} x = -\operatorname{sind} x \tag{3.50}$$

3.2.9. Gamma and Related Functions

Here, $h = 1$. The labels (HMF x) refers to the number in the *Handbook of Mathematical Functions* where special functions have been defined. The function $G(x)$ is defined by formula 2.50.

$$\Delta\Gamma(x) = (x-1)\Gamma(x) \tag{3.51}$$
$$\Delta G(x) = \Gamma(x) \tag{3.52}$$
$$\Delta x G(x) = G(x+2) \tag{3.53}$$
$$\Delta B(x,a) = \frac{-a}{x+a} B(x,a) \tag{3.54}$$
$$\Delta\psi_0(x) = \frac{1}{x} \tag{3.55}$$
$$\Delta\psi_0'(x) = -\frac{1}{x^2} \tag{3.56}$$
$$\Delta\psi_n(x) = \frac{(-1)^n n!}{x^{n+1}} \tag{3.57}$$
$$\Delta\Gamma_q = \frac{q-q^x}{1-q}\Gamma_q(x) \qquad \text{for } 0 < q < 1 \qquad \text{(NHMF 5.18.4)} \tag{3.58}$$
$$\Delta P(x,a) = \frac{-a^x e^{-a}}{\Gamma(x+1)} \qquad \text{(HMF 6.5.1)} \tag{3.59}$$
$$\Delta\gamma(x,a) = (x-1)\gamma(x,a) - a^x e^a \qquad \text{(HMF 6.5.2)} \tag{3.60}$$
$$\Delta\Gamma(x,a) = (x-1)\Gamma(x,a) + a^x e^a \qquad \text{(HMF 6.5.3)} \tag{3.61}$$
$$\Delta\gamma^*(x,a) = \gamma^*(x,a) - \frac{1}{e^a\Gamma(x+1)} \qquad \text{(HMF 6.5.4)} \tag{3.62}$$
$$\Delta I_t(x,a) = (t-1)I_t(x,a) + (t-1)I_t(x,a-1) \qquad \text{(HMF 6.6.2)} \tag{3.63}$$

3.2.10. More Exponential Forms

$$\Delta 2^x \frac{a^{2^x}+1}{a^{2^x}-1} = 2^x \frac{a^{2^x}-1}{a^{2^x}+1} \tag{3.64}$$

3.2.11. The Discrete Square Root

$$\Delta x^{\left\langle \frac{1}{2} \right\rangle} = \frac{1}{2} x^{\left\langle -\frac{1}{2} \right\rangle} = \frac{1}{2\left(x+\frac{1}{2}\right)^{\left\langle \frac{1}{2} \right\rangle}} \tag{3.65}$$

$$\Delta \left(x^{\left\langle \frac{1}{2} \right\rangle}\right)^2 = \left(x^{\left\langle -\frac{1}{2} \right\rangle}\right)^2 \left(x + \frac{3}{4}\right) \tag{3.66}$$

$$\Delta \left(x^{\left\langle \frac{1}{2} \right\rangle}\right)^3 = \left(x^{\left\langle -\frac{1}{2} \right\rangle}\right)^3 \left(\frac{3}{2}x^2 + \frac{9}{4}x + \frac{7}{8}\right) \tag{3.67}$$

$$\Delta \left(x^{\left\langle \frac{1}{2} \right\rangle}\right)^4 = \left(x^{\left\langle -\frac{1}{2} \right\rangle}\right)^4 \left(2x^3 + \frac{7}{2}x^2 + \frac{9}{2}x + \frac{15}{16}\right) \tag{3.68}$$

$$\Delta \left(x^{\left\langle \frac{1}{2} \right\rangle}\right)^n = \left(x^{\left\langle -\frac{1}{2} \right\rangle}\right)^n \sum_{k=0}^{n-1} \binom{n}{k} \frac{2^{n-k}-1}{2^{n-k}} x^k \tag{3.69}$$

3.2.12. Forms involving $\sin x$ and 2^x.

$$\Delta 2^x \sin\left(\frac{a}{2^x}\right) = \sin\left(\frac{a}{2^{x+1}}\right)\sin^2\left(\frac{a}{2^{x+2}}\right) \tag{3.70}$$

$$\Delta 2^x \sin^2\left(\frac{a}{2^x}\right) = 2^{2x+2}\sin^4\left(\frac{a}{2^{x+1}}\right) \tag{3.71}$$

$$\Delta \tan\left(\frac{a}{2^x}\right) = -\frac{\tan\left(\frac{a}{2^{x+1}}\right)}{\cos\left(\frac{a}{2^x}\right)} \tag{3.72}$$

$$\Delta(-2)^x \sin\left(\frac{a}{2^x}\right) = (-2)^{x+3}\sin\left(\frac{a}{2^{x+1}}\right)\cos\left(\frac{a}{2^{x+1}}\right) \tag{3.73}$$

$$\Delta\frac{1}{2^x\sin\left(\frac{a}{2^x}\right)} = -2\frac{\sin^2\left(\frac{a}{2^{x+2}}\right)}{2^x\sin\left(\frac{a}{2^x}\right)} \tag{3.74}$$

$$\Delta\frac{1}{2^x\tan\left(\frac{a}{2^x}\right)} = \frac{\tan\left(\frac{a}{2^{x+1}}\right)}{2^{x+1}} \tag{3.75}$$

$$\Delta\left(\frac{1}{2^x\tan\left(\frac{a}{2^x}\right)}\right)^2 = \frac{2}{2^x} - \left(\frac{\tan\left(\frac{a}{2^{x+1}}\right)}{2^{x+1}}\right) \tag{3.76}$$

$$\Delta\frac{\cos\left(\frac{a}{2^x}\right)}{2^x\sin\left(\frac{a}{2^x}\right)} = \frac{\sin\left(\frac{a}{2^x}\right)}{\left(2^{x+1}\cos\frac{a}{2^{x+1}}\right)} \tag{3.77}$$

$$\Delta\cot(2^x a) = -\frac{1}{\sin\left(2^{x+1}a\right)} \tag{3.78}$$

$$\Delta\frac{\ln(2\sin 2^x a)}{2^x} = -\frac{\log\tan 2^x a}{2^{x+1}} \tag{3.79}$$

$$\Delta\frac{1}{\left(2^x\sin\left(\frac{a}{2^x}\right)\right)^2} = -\frac{1}{\left(2^{x+1}\cos\left(\frac{a}{2^{x+1}}\right)\right)^2} \tag{3.80}$$

$$\Delta\frac{1}{\prod_{k=0}^{2n+1}\cos(a(x+k))} = 2\sin(a(n+1))\frac{\sin(a(x+n+1))}{\prod_{k=0}^{2n+2}\cos(a(x+k))} \tag{3.81}$$

$$\Delta\frac{(-1)^x}{\sin(ax)\sin(a(x+1))} = 2\cos a\frac{(-1)^{x+1}}{\sin(ax)\sin(a(x+2))} \tag{3.82}$$

$$\Delta\frac{(-1)^x}{\prod_{k=0}^{2n+1}\sin(a(x+k))} = 2\cos(a(n+1))\frac{(-1)^{x+1}\sin(a(x+n+1)}{\prod_{k=0}^{2n+2}\sin(a(x+k))} \tag{3.83}$$

$$\Delta\frac{(-1)^x}{\cos(ax)\cos(a(x+1))} = 2\cos a\frac{(-1)^{x+1}}{\cos(ax)\cos(a(x+2))} \tag{3.84}$$

$$\Delta\frac{(-1)^x}{\prod_{k=0}^{2n+1}\cos(a(x+k))} = 2\cos(a(n+1))\frac{(-1)^{x+1}\cos(a(x+2n+1))}{\prod_{k=0}^{2n+2}\cos(a(x+k))} \tag{3.85}$$

3.2.13. Forms involving $\operatorname{atan} x$

$$\Delta\operatorname{atan}\left(\frac{ax+b}{cx+d}\right) = \operatorname{atan}\left(\frac{bc-ad}{a^2+ab+c^2+cd+(2ab+b^2+2cd+d^2)x+(b^2+d^2)x^2}\right) \tag{3.86}$$

$$\Delta\operatorname{atan}(xf(x)) = \operatorname{atan}\left(\frac{\Delta f(x)}{1+f(x)f(x+1)}\right) \tag{3.87}$$

$$\Delta 2^x\operatorname{atan}\left(\frac{a}{2^x}\right) = 2^x\operatorname{atan}\left(\frac{a^3}{2^{3x+2}+3a^2 2^x}\right) \tag{3.88}$$

3.3. Sums

3.3.1. Basics

$$\sum a = ahx + C \tag{3.89}$$

$$\sum (-1)^x \binom{n}{x} = (-1)^{x+1}\binom{n-1}{x-1} + C \tag{3.90}$$

$$\sum x = \frac{1}{2}x^2 - \frac{h}{2}x + C \tag{3.91}$$

$$\sum x^2 = \frac{1}{3}x^3 - \frac{2h}{3}x^2 - \frac{h^2}{6}x + C \tag{3.92}$$

$$\sum x^3 = \frac{1}{4}x^4 - \frac{h}{2}x^3 + \frac{h^2}{4}x^2 + C \tag{3.93}$$

$$\sum x^4 = \frac{1}{5}x^5 - \frac{h}{2}x^4 + \frac{h^2}{3}x^3 - \frac{h^4}{30}x + C \tag{3.94}$$

$$\sum a^x = \frac{h}{a^h - 1}a^x + C \tag{3.95}$$

$$\sum \log_a x = \log_a \Gamma(x) + C \qquad h = 1 \tag{3.96}$$

$$?\sum x^n = \sum_{k=1}^{n} \frac{S_1(n,k)}{k+1} x^{\langle k+1\rangle} + C \qquad h = 1 \tag{3.97}$$

$$\sum x^n = \frac{B_{n+1}(x)}{n+1} + C \qquad h = 1 \tag{3.98}$$

$$\sum x^n = \frac{E_n(x)}{2} + C \qquad h = 1 \tag{3.99}$$

$$\sum x^{\langle a\rangle} = \frac{x^{\langle a+1\rangle}}{n+1} + C \qquad h = 1, a \neq -1 \tag{3.100}$$

$$\sum \frac{1}{x} = \psi_0(x) + C \qquad h = 1 \tag{3.101}$$

$$\sum \frac{1}{x^n} = \frac{(-1)^{n-1}\psi_{n-1}(x)}{(n-1)!} + C \qquad h = 1 \tag{3.102}$$

3.3.2. Trigonometric Functions

$$\sum \sin(2ax) = -\frac{\cos[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.103}$$

$$\sum \sin(2\pi nx) = x\sin(2\pi nx) \tag{3.104}$$

$$\sum \cos(2ax) = \frac{\sin[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.105}$$

$$\sum \cos(2\pi nx) = x\cos(2\pi nx) \tag{3.106}$$

$$\sum \sin^2(ax) = \frac{x}{2} - \frac{\sin[a(2x-1)]}{4\sin a} \qquad a \neq \pi n \tag{3.107}$$

$$\sum \sin^2(\pi nx) = x\sin^2(2\pi nx) \tag{3.108}$$

$$\sum \cos^2(ax) = \frac{x}{2} + \frac{\sin[a(2x-1)]}{4\sin a} \qquad a \neq \pi n \tag{3.109}$$

$$\sum \cos^2(\pi nx) = x\cos^2(2\pi nx) \tag{3.110}$$

$$\sum x\sin(2ax) = \frac{\sin[a(2x-1)]}{4\sin^2 a} - x\frac{\cos[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.111}$$

$$\sum x\sin(2\pi nx) = \frac{1}{2}x(x-1)\sin^2(2\pi nx) \tag{3.112}$$

$$\sum x\cos(2ax) = \frac{\cos[a(2x-1)]}{4\sin^2 a} + x\frac{\sin[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.113}$$

$$\sum x\cos(2\pi nx) = \frac{1}{2}x(x-1)\cos^2(2\pi nx) \tag{3.114}$$

$$\sum (-1)^x \cos(2bx) = (-1)^{x+1}\frac{\cos[b(2x-1)]}{2\cos b} \qquad b \neq \pi\left(n+\frac{1}{2}\right) \tag{3.115}$$

$$\sum (-1)^x \cos\left(2\pi\left(n+\frac{1}{2}\right)x\right) = (-1)^x x\cos\left(2\pi\left(n+\frac{1}{2}\right)x\right) \tag{3.116}$$

$$\sum (-1)^x \sin(2bx) = (-1)^{x+1}\frac{\sin[b(2x-1)]}{2\cos b} \qquad b \neq \pi\left(n+\frac{1}{2}\right) \tag{3.117}$$

$$\sum (-1)^x \sin\left(2\pi\left(n+\frac{1}{2}\right)x\right) = (-1)^x x\sin\left(2\pi\left(n+\frac{1}{2}\right)x\right) \tag{3.118}$$

$$\sum a^x \sin(bx) = a^x\frac{a\sin[b(x-1)] - \sin(bx)}{a^2 - 2a\cos b + 1} \qquad a > 0, \quad a \neq 1 \tag{3.119}$$

$$\sum a^x \cos(bx) = a^x\frac{a\cos[b(x-1)] - \cos(bx)}{a^2 - 2a\cos b + 1} \qquad a > 0, \quad a \neq 1 \tag{3.120}$$

$$\sum \tan\pi x = x\tan\pi x \tag{3.121}$$

$$\sum \cot\pi x = x\cot\pi x \tag{3.122}$$

3.3.3. Hyperbolic Functions

$$\sum \sinh(2ax) = \frac{\cosh\left[a(2x-1)\right]}{2\sinh(a)} \qquad a \neq \pi n \qquad (3.123)$$

$$\sum \sinh(2\pi n x) = x\sinh(2\pi n x) \qquad (3.124)$$

$$\sum \cosh(2ax) = \frac{\sinh\left[a(2x-1)\right]}{2\sinh(a)} \qquad a \neq \pi n \qquad (3.125)$$

$$\sum \cosh(2\pi n x) = x\cosh(2\pi n x) \qquad (3.126)$$

$$\sum \sinh^2(ax) = \frac{\sinh\left[a(2x-1)\right]}{4\sinh(a)} - \frac{x}{2} \qquad a \neq \pi n \qquad (3.127)$$

$$\sum \sinh^2(\pi n x) = x\sinh^2(2\pi n x) \qquad (3.128)$$

$$\sum \cosh^2(ax) = \frac{\sinh\left[a(2x-1)\right]}{4\sinh(a)} + \frac{x}{2} \qquad a \neq \pi n \qquad (3.129)$$

$$\sum \cosh^2(\pi n x) = x\cosh^2(2\pi n x) \qquad (3.130)$$

$$\sum x\sinh(2ax) = \frac{\sinh[a(2x-1)]}{4\sinh^2 a} - x\frac{\cosh[a(2x-1)]}{2\sinh a} \qquad a \neq \pi n \qquad (3.131)$$

$$\sum x\sinh(2\pi n x) = \frac{1}{2}x(x-1)\sinh^2(2\pi n x) \qquad (3.132)$$

$$\sum x\cosh(2ax) = \frac{\cosh[a(2x-1)]}{4\sinh^2 a} + x\frac{\sinh[a(2x-1)]}{2\sinh a} \qquad a \neq \pi n \qquad (3.133)$$

$$\sum x\cosh(2\pi n x) = \frac{1}{2}x(x-1)\cosh^2(2\pi n x) \qquad (3.134)$$

$$\sum (-1)^x \cosh(2bx) = (-1)^x \frac{\cosh[b(2x-1)]}{2\cosh b} \qquad (3.135)$$

$$?\sum (-1)^x \sinh(2bx) = (-1)^x \frac{\sinh[b(2x-1)]}{2\cosh b} \qquad (3.136)$$

$$\sum a^x \sinh(bx) = a^x \frac{a\sinh[b(x-1)] - \sinh(bx)}{a^2 - 2a\cosh b + 1} \qquad a > 0, \quad a \neq 1 \qquad (3.137)$$

$$\sum a^x \cosh(bx) = a^x \frac{a\cosh[b(x-1)] - \cosh(bx)}{a^2 - 2a\cosh b + 1} \qquad a > 0, \quad a \neq 1 \qquad (3.138)$$

$$\sum \tanh \pi i x = (x-1)\tanh \pi i x \qquad (3.139)$$

$$\sum \coth \pi i x = (x-1)\coth \pi i x \qquad (3.140)$$

3.3.4. Exponential Sums

$$\sum \frac{2^x}{x} = \mathrm{Bs}(x) \qquad (3.141)$$

$$\sum \frac{\operatorname{sind} x}{x} = \mathrm{Ss}(x) \qquad (3.142)$$

$$\sum \frac{\operatorname{cosd} x}{x} = \mathrm{Cs}(x) \qquad (3.143)$$

$$\sum \frac{\operatorname{sinhd} x}{x} = \mathrm{Shs}(x) \qquad (3.144)$$

$$\sum \frac{\operatorname{coshd} x}{x} = \mathrm{Chs}(x) \qquad (3.145)$$

3.3.5. Gamma Functions

$$\sum \Gamma(x) = G(x) + C \tag{3.146}$$

$$\sum G(x) = (x-2)G(x-2) + C \tag{3.147}$$

$$\sum \psi_0(x) = (x-1)\psi_0(x) - x + C \tag{3.148}$$

$$\sum \psi_0(-x) = x\psi_0(-x) - x \tag{3.149}$$

$$\sum \psi_0(mx) = (\ln m - 1)x - \frac{m-1}{2m} + \frac{1}{m}\sum_{k=0}^{m-1}\left(x - 1 + \frac{k}{m}\right)\psi_0\left(x - \frac{k}{m}\right) \tag{3.150}$$

$$\sum \psi_1(x) = (x-1)\psi_1(x) + \psi_0(x) + C \tag{3.151}$$

$$\sum \psi_n(x) = (x-1)\psi_n(x) + (-1)^{n-1}\psi_{n-1}(x) + C \tag{3.152}$$

3.3.6. Forms involving $ax + b$

$$\sum \frac{1}{ax+b} = \frac{1}{a}\psi_0\left(x + \frac{b}{a}\right) \tag{3.153}$$

$$\sum \frac{x}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\psi_0\left(x + \frac{b}{a}\right) \tag{3.154}$$

$$\sum \frac{x^2}{ax+b} = \frac{ax^2 - (2b+1)x}{2a^2} + \frac{b^2}{a^3}\psi_0\left(x + \frac{b}{a}\right) \tag{3.155}$$

$$\sum \frac{1}{x(ax+b)} = \frac{1}{b}\psi_0(x) - \frac{1}{b}\psi_0\left(x + \frac{b}{a}\right) \tag{3.156}$$

3.3.7. Forms involving $x^2 + a^2$

$$\sum \frac{1}{x^2+a^2} = \frac{1}{2ai}\left(\psi_0(x-ia) - \psi_0(x+ia)\right) \tag{3.157}$$

$$\sum \frac{x}{x^2+a^2} = \frac{1}{2}\left(\psi_0(x-ia) + \psi_0(x+ia)\right) \tag{3.158}$$

$$\sum \frac{x^2}{x^2+a^2} = x - \frac{a}{2i}\left(\psi_0(x-ia) - \psi_0(x+ia)\right) \tag{3.159}$$

$$\sum \frac{x^3}{x^2+a^2} = \frac{x^{\langle 2\rangle}}{2} - \frac{a^2}{2}\left(\psi_0(x-ia) + \psi_0(x+ia)\right) \tag{3.160}$$

$$\sum \frac{1}{x(x^2+a^2)} = \frac{1}{a^2}\psi_0(x) - \frac{1}{2a^2}(\psi_0(x-ai) + \psi_0(x+ai)) \tag{3.161}$$

$$\sum \frac{1}{x^2(x^2+a^2)} = -\psi_1(x) - \frac{1}{2ai}\left(\psi_0(x-ia) - \psi_0(x+ia)\right) \tag{3.162}$$

$$\sum \frac{1}{x^3(x^2+a^2)} = \frac{1}{2}\psi_2(x) - \psi_0(x) + \frac{1}{4ai}\left(\psi_0(x-ia) - \psi_0(x+ia)\right) \tag{3.163}$$

In these identities, the right hand side is real-valued if x is real.

3.3.8. Discrete Square Root

$$\sum \frac{1}{x^{\langle \frac{1}{2} \rangle}} = 2\left(x + \frac{1}{2}\right) + C \tag{3.164}$$

$$\sum \frac{x}{x^{\langle \frac{1}{2} \rangle}} = 2\left(x + \frac{1}{2}\right)^{\langle \frac{3}{2} \rangle} - \frac{4}{15} x^{\langle \frac{5}{2} \rangle} \tag{3.165}$$

3.4. Taylor Series

The formal discrete Taylor series of a function f is given by:

$$f^*(x) = \sum_{k=0}^{\infty} f^{\langle n \rangle}(a) \frac{(x-a)^{\langle n \rangle}}{n!}. \tag{3.166}$$

We write $f(x) \sim f^*(x)$ to denote that f^* is the formal Taylor series of f.

Example Let us find the discrete Taylor series of $f(x) = 2^x$ at $x = 0$. Since, $f^{\langle n \rangle}(x) = 2^x$, we have $f^{\langle n \rangle}(0) = 1$. Thus, the Taylor series is given by

$$2^x \sim \sum_{k=0}^{\infty} \frac{x^{\langle n \rangle}}{n!}$$

Graphs for the truncated series is given in Figure 3.1. From the graphs it looks like $f^*(x)$ only converges for $x > 0$.

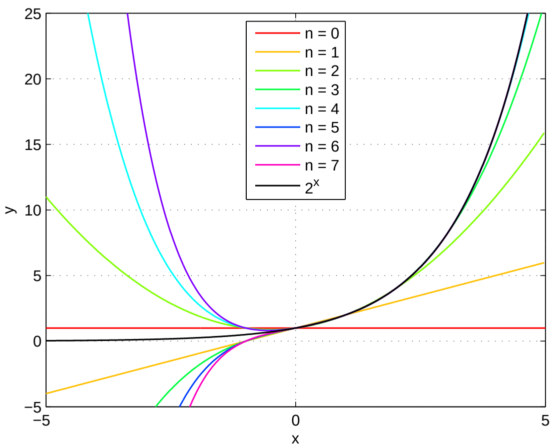


Figure 3.1.: Taylor series for 2^x

$$x^n \sim \sum_{k=0}^{n} S_1(n,k) x^{\langle k \rangle} \tag{3.167}$$

$$a^x \sim \sum_{k=0}^{\infty} (a-1)^k \frac{x^{\langle k \rangle}}{k!} \tag{3.168}$$

$$\log_a(x+1) \sim \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} \log_a(m+1) \frac{x^{\langle k \rangle}}{k!} \tag{3.169}$$

Here we define:

$$\mathrm{soc}_k(x) = \begin{cases} \sin(x) & k \text{ is even} \\ \cos(x) & k \text{ is odd} \end{cases}$$

$$\sin(x) \sim \sum_{k=0}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor} \sin^k\left(\frac{1}{2}\right) \mathrm{soc}_k\left(\frac{n}{2}\right) \frac{x^{\langle k \rangle}}{k!} \tag{3.170}$$

$$\cos(x) \sim \sum_{k=0}^{\infty} (-1)^{\lfloor \frac{k+1}{2} \rfloor} \sin^k\left(\frac{1}{2}\right) \mathrm{soc}_{k+1}\left(\frac{n}{2}\right) \frac{x^{\langle k \rangle}}{k!} \tag{3.171}$$

$$\mathrm{sind}(x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{x^{\langle 2k+1 \rangle}}{(2k+1)!} \tag{3.172}$$

$$\mathrm{cosd}(x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{x^{\langle 2k \rangle}}{(2k)!} \tag{3.173}$$

$$\Gamma(x+1) \sim \sum_{k=0}^{\infty} !(k+1) \frac{x^{\langle k \rangle}}{k!} \qquad !n = 1, 0, 1, 2, 9, 44, 265, 1854 \tag{3.174}$$

$$G(x+1) \sim \sum_{k=0}^{\infty} !k \frac{x^{\langle k \rangle}}{k!} \qquad !n = 1, 0, 1, 2, 9, 44, 265, 1854 \tag{3.175}$$

$$\psi_0(x+1) \sim \gamma + \sum_{k=1}^{\infty} (-1)^k (k-1)! x^{\langle k \rangle} \tag{3.176}$$

$$\frac{1}{x+1} \sim \sum_{k=0}^{\infty} (-1)^{k+1} (k+1)! x^{\langle k \rangle} \tag{3.177}$$

$$\mathrm{Bs}(x+1) \sim \psi_0(x+1) + \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{x^{\langle k \rangle}}{k!} \tag{3.178}$$

$$= \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k (k!)^2 + 1}{k \cdot k!} x^{\langle k \rangle} \tag{3.179}$$

$$\mathrm{Ss}(x+1) \sim \psi_0(x+1) + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k+1} \cdot \frac{x^{\langle 2k+1 \rangle}}{(2k+1)!} \tag{3.180}$$

$$= \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{(2k+1) \cdot (2k+1)!(k-1)! + 1}{(2k+1) \cdot (2k+1)!} x^{\langle k \rangle} \tag{3.181}$$

$$\mathrm{Cs}(x+1) \sim \psi_0(x+1) + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k} \cdot \frac{x^{\langle 2k \rangle}}{(2k)!} \tag{3.182}$$

$$= \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{2k \cdot (2k)!(k-1)! + 1}{2k \cdot (2k)!} x^{\langle k \rangle} \tag{3.183}$$

$$\tag{3.184}$$

3.5. Miscellaneous Difference Identities

$$\Delta \mathcal{F}(x) = \mathcal{F}(x-1) \tag{3.185}$$

$$\Delta W(x) = \Delta \ln x - \Delta \ln W(x) \tag{3.186}$$

3.6. Difference Equations

3.6.1. $p(x)f(x+1) - f(x) = q(x)$

$$f(x) = \frac{\sum(q(x)\prod p(x))}{\prod p(x)}$$

3.6.2. $f(x)\Delta f(x) = 1$

$$f(x) = \frac{c\mathcal{F}(x) - \mathcal{F}(x-1)}{c\mathcal{F}(x-1) - \mathcal{F}(x-2)}$$

4. Sum (xh)

4.1. Definitions and Properties

4.1.1. Definitions

$$\mathrm{P}\, f(x) = f(xh) - f(x) \qquad h \neq 0 \tag{4.1}$$
$$\mathrm{P}\,\mathrm{P}^{-1} f(x) = f(x) \tag{4.2}$$

Thus, $\mathrm{P}^{-1} f(x)$ is any function $F(x)$ that satisfies $\mathrm{P}\, F(x) = f(x)$.

4.1.2. Properties of Operators

$$\mathrm{P}(af(x) + bg(x)) = a\,\mathrm{P}\, f(x) + b\,\mathrm{P}\, g(x) \tag{4.3}$$
$$\mathrm{P}(f(x)g(x)) = f(hx)\,\mathrm{P}\, g(x) + g(x)\,\mathrm{P}\, f(x) \tag{4.4}$$
$$\sum_{a}^{b-1} f(h^{x-a}x) = \mathrm{P}^{-1} f(x)\Big|_a^b \tag{4.5}$$
$$\mathrm{P}^{-1} f(x)g(x) = f(x)\,\mathrm{P}^{-1} g(x) - \mathrm{P}^{-1}\,\mathrm{P}\, f(x)\,\mathrm{P}^{-1} g(x+1) \tag{4.6}$$
$$\mathrm{P}^{-1} xf(x) = xD\,\mathrm{P}^{-1} \int f(x) \tag{4.7}$$

4.1.3. Analogues of Elementary Functions

Elementary Function	Analog
x^a	$(\log_h x)^{\langle a \rangle}$
e^x	-
$\sin x$	-
$\cos x$	-

4.2. Differences

4.2.1. Basics

$$\mathrm{P}\, c = 0 \tag{4.8}$$
$$\mathrm{P}\, x^a = (h^a - 1)x^a \tag{4.9}$$
$$\mathrm{P} \log_a x = \log_a h \tag{4.10}$$
$$\mathrm{P}(\log_h x)^{\langle n \rangle} = n(\log_h x)^{\langle n-1 \rangle} \tag{4.11}$$
$$\mathrm{P}\, a^x = a^x(a^{(h-1)x} - 1) \tag{4.12}$$
$$\mathrm{P}\, 2^{\log_h x} = 2^{\log_h x} \tag{4.13}$$

4.3. Sums

4.3.1. Basics

$$\mathrm{P}^{-1}\, a = a \log_h x \tag{4.14}$$

$$\mathrm{P}^{-1}\, x^a = \frac{x^a}{h^a - 1} \tag{4.15}$$

$$\mathrm{P}^{-1} \log_a x = \frac{\log_a x(\log_h x - 1)}{2} \tag{4.16}$$

$$\mathrm{P}^{-1}(\log_h x)^{\langle n\rangle} = \frac{(\log_h x)^{\langle n+1\rangle}}{n+1} \tag{4.17}$$

$$\mathrm{P}^{-1}(\log_a x)^{\langle n\rangle} = \frac{\log_a x(\log_h x - 1)^{\langle n\rangle}}{n+1} \tag{4.18}$$

4.3.2. Trigonometric Functions

$$\mathrm{P}^{-1} \sin(x) = S(x,h) = \sum_{k=1}^{\infty} \sin(h^{-k}x) \tag{4.19}$$

$$\mathrm{P}^{-1} \cos(x) = C(x,h) = \log_h x + \sum_{k=1}^{\infty}(\cos(h^{-k}x) - 1) \tag{4.20}$$

$$\mathrm{P}^{-1}\, x\cos(x) = xS'(x,h) \tag{4.21}$$

$$\mathrm{P}^{-1}\, x\sin(x) = -xC'(x,h) \tag{4.22}$$

$$\tag{4.23}$$

5. Product $(x+1)$

5.1. Definitions and Properties

5.1.1. Definition

$$\operatorname{Q} f(x) = \frac{f(x+1)}{f(x)} \tag{5.1}$$

5.1.2. Properties of Operators

$$\operatorname{Q}(f(x)^a g(x)^b) = (\operatorname{Q} f(x))^a (\operatorname{Q} g(x))^b \tag{5.2}$$

$$\operatorname{Q} f(x)^{g(x)} = \left[\operatorname{Q} e^{g(x)}\right]^{\ln f(x+1)} [\operatorname{Q} f(x)]^{g(x)} \tag{5.3}$$

$$\operatorname{Q} f(x)^{g(x)} = \left[\operatorname{Q} e^{g(x)}\right]^{\ln f(x)} [\operatorname{Q} f(x)]^{g(x+1)} \tag{5.4}$$

$$\operatorname{Q} \prod f(x) = f(x) \tag{5.5}$$

$$\prod_a^{b-1} f(x) = \frac{F(b)}{F(a)} \qquad F(x) = \prod f(x) \tag{5.6}$$

$$\prod f(x)^{g(x)} = \frac{(\prod f(x))^{g(x)}}{\prod(\prod f(x+1))^{\Delta g(x)}} \tag{5.7}$$

5.1.3. Relation with Additive Operators

$$a^{\Delta f(x)} = \operatorname{Q} a^{f(x)} \tag{5.8}$$

$$a^{\sum f(x)} = \prod a^f(x) \tag{5.9}$$

5.1.4. Analogues of Elementary Functions

Elementary Function	Analog
x^n	$\exp\left(x^{\langle n\rangle}\right)$
x^{-n}	$\exp\left(x^{\langle -n\rangle}\right)$
e^x	$\exp(2^x)$
$\sin x$	$\exp(\operatorname{sind} x)$
$\cos x$	$\exp(\operatorname{cosd} x)$

5.2. Quotients

5.2.1. Basics

$$\mathrm{Q}\, c = 1 \tag{5.10}$$

$$\mathrm{Q}\, x = 1 + \frac{1}{x} \tag{5.11}$$

$$\mathrm{Q}\, x^2 = 1 + \frac{2}{x} + \frac{1}{x^2} \tag{5.12}$$

$$\mathrm{Q}\, x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{x^k} \tag{5.13}$$

$$\mathrm{Q}\, x^{\langle a \rangle} = \frac{x+1}{x-a+1} \tag{5.14}$$

$$\mathrm{Q}\, a^x = a \tag{5.15}$$

$$\mathrm{Q}\, a^{2^x} = a^{2^x} \tag{5.16}$$

$$\mathrm{Q}\, a^{b^x} = a^{b^x(b-1)} \tag{5.17}$$

$$\mathrm{Q} \log_a x = \log_x (x+1) \tag{5.18}$$

$$\mathrm{Q} \exp\left(x^{\langle n \rangle}\right) = \exp\left(n x^{\langle n-1 \rangle}\right) \tag{5.19}$$

5.2.2. Trigonometric Functions

$$\mathrm{Q} \sin x = \cos 1 + \sin 1 \cot x \tag{5.20}$$

$$\mathrm{Q} \cos x = \cos 1 - \sin 1 \tan x \tag{5.21}$$

$$\mathrm{Q} \tan x = \frac{\cos 1 + \sin 1 \cot x}{\cos 1 - \sin 1 \tan x} \tag{5.22}$$

$$\mathrm{Q} \cot x = \frac{\cos 1 - \sin 1 \tan x}{\cos 1 + \sin 1 \cot x} \tag{5.23}$$

$$\mathrm{Q} \sec x = \frac{1}{\cos 1 - \sin 1 \tan x} \tag{5.24}$$

$$\mathrm{Q} \csc x = \frac{1}{\cos 1 + \sin 1 \cot x} \tag{5.25}$$

$$\tag{5.26}$$

5.2.3. Hyperbolic Functions

$$\mathrm{Q} \sinh x = \cosh 1 + \sinh 1 \coth x \tag{5.27}$$

$$\mathrm{Q} \cosh x = \cosh 1 + \sinh 1 \tanh x \tag{5.28}$$

$$\mathrm{Q} \tanh x = \frac{\cosh 1 + \sinh 1 \coth x}{\cosh 1 + \sinh 1 \tanh x} \tag{5.29}$$

$$\mathrm{Q} \coth x = \frac{\cosh 1 + \sinh 1 \tanh x}{\cosh 1 + \sinh 1 \coth x} \tag{5.30}$$

$$\mathrm{Q}\, \mathrm{sech}\, x = \frac{1}{\cosh 1 + \sinh 1 \tanh x} \tag{5.31}$$

$$\mathrm{Q}\, \mathrm{csch}\, x = \frac{1}{\cosh 1 + \sinh 1 \coth x} \tag{5.32}$$

$$\tag{5.33}$$

5.2.4. Discrete Additive Trigonometric Functions

$$\mathrm{Q}\,\mathrm{sind}\,x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cot\left(\frac{\pi x}{4}\right) \tag{5.34}$$

$$\mathrm{Q}\,\mathrm{cosd}\,x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tan\left(\frac{\pi x}{4}\right) \tag{5.35}$$

5.2.5. Gamma and Related Functions

$$\mathrm{Q}\,\Gamma(x) = x \tag{5.36}$$

$$\mathrm{Q}\,\Gamma^n(x) = x^n \tag{5.37}$$

$$\mathrm{Q}(a^x\Gamma^n(x)) = ax^n \tag{5.38}$$

$$\mathrm{Q}\,\Gamma(x+a) = x+a \tag{5.39}$$

$$\mathrm{Q}\,B(a,x) = \frac{1}{x+a}\left(\frac{x\Gamma(x)+\Gamma(a)}{\Gamma(x)+\Gamma(a)}\right) \tag{5.40}$$

5.3. Products

5.3.1. Basics

$$\prod(ax^n) = (a^x\Gamma^n(x)) \tag{5.41}$$

$$\prod(ax+b) = a^x\Gamma\left(x+\frac{b}{a}\right) \tag{5.42}$$

$$\prod a^x = a^{x(x-1)/2} \tag{5.43}$$

$$\prod a\frac{(x-p_1)(x-p_2)\cdots(x-p_m)}{(x-q_1)(x-q_2)\cdots(x-q_n)} = a^x\frac{\Gamma(x-p_1)\Gamma(x-p_2)\cdots\Gamma(x-p_m)}{\Gamma(x-q_1)\Gamma(x-q_2)\cdots\Gamma(x-q_n)} \tag{5.44}$$

$$\prod a^{\frac{1}{x}} = a^{\psi_0(x)} \tag{5.45}$$

5.4. Taylor Series

$$f(x) = \prod_{n=0}^{\infty} f^{\mathrm{Q}\,n}(0)\exp\left(\frac{x^{\langle n\rangle}}{n!}\right) \tag{5.46}$$

5.5. Miscellaneous Identities

$$\mathrm{Q}\,\mathcal{F}(x) = k + \mathrm{Q}\,\mathcal{F}(x-k) \tag{5.47}$$

5.6. Quotient Equations

The solution of the equation:

$$\Delta \frac{f(x+1)^{p(x)}}{f(x)} = q(x) \tag{5.48}$$

is given by

$$f(x) = \left(\prod q(x)^{\prod p(x)}\right)^{1/\prod p(x)}$$

6. Product (hx)

6.1. Definition

$$\mathrm{Q}\, f(x) = \frac{f(xh)}{f(x)}$$

6.2. Quotients

6.2.1. Basics

$$\mathrm{Q}\, c = 1 \tag{6.1}$$
$$\mathrm{Q}\, x = h \tag{6.2}$$
$$\mathrm{Q}\, x^n = h^n \tag{6.3}$$
$$\mathrm{Q}\, a^x = a^{x(h-1)} \tag{6.4}$$
$$\mathrm{Q}\log_a x = 1 + \log_x h \tag{6.5}$$
$$\mathrm{Q}\log_x a = 1 - \frac{\ln h}{\ln h + \ln x} \tag{6.6}$$

7. Generalised Operators

7.1. Definitions

$$\Delta_g f(x) = f(g(x)) - f(x) \tag{7.1}$$

$$\mathrm{Q}_g\, f(x) = \frac{f(g(x))}{f(x)} \tag{7.2}$$

7.2. Theorems

$$\Delta_g(af_1(x) + bf_2(x)) = a\Delta_g f_1(x) + b\Delta_g f_2(x) \quad \text{Linearity} \tag{7.3}$$

$$\Delta_g(f_1(x)f_2(x)) = f_1(g(x))\Delta_g f_2(x) + f_2(x)\Delta_g f_1(x) \quad \text{Product Rule} \tag{7.4}$$

$$\Delta_g(f_1(x)f_2(x)) = f_1(x)\Delta_g f_2(x) + f_2(g(x))\Delta_g f_1(x) \quad \text{Product Rule} \tag{7.5}$$

$$\sum_{k=0}^{n-1} \Delta_g f(g^k(x)) = f(g^n(x)) - f(x) \quad \text{Fundemental Theorem} \tag{7.6}$$

$$\left(\prod_g g'(x)\right) D\Delta_g f(x) = \Delta_g \left(f'(x)\prod_g g'(x)\right) \tag{7.7}$$

7.3. Standard Functions of Difference Operators

7.4. Definitions

$$I_\Delta g(x) = f(x) \Rightarrow f(g(x)) - f(x) = 1 \tag{7.8}$$

7.5. Theorems

$$\Delta_g C(I_\Delta g(x)) = 0 \quad \text{for any } C(x+1) = C(x) \tag{7.9}$$

$$\Delta_g 2^{I_\Delta g(x)} = 2^{I_\Delta g(x)} \tag{7.10}$$

$$\Delta_g \left(I_\Delta g(x)\right)^{\langle n\rangle} = n\left(I_\Delta g(x)\right)^{\langle n-1\rangle} \tag{7.11}$$

$$\Delta_g \operatorname{sind}\left(I_\Delta g(x)\right) = \operatorname{cosd}\left(I_\Delta g(x)\right) \tag{7.12}$$

$$\Delta_g \operatorname{cosd}\left(I_\Delta g(x)\right) = -\operatorname{sind}\left(I_\Delta g(x)\right) \tag{7.13}$$

$$\tag{7.14}$$

$$p(x+1) = g(p(x)) \Rightarrow I_\Delta g(x) = p^{-1}(x) \tag{7.15}$$

$$p(x) = I_\Delta g(x) \Rightarrow f(g(x)) = f(p(p^{-1}x) + k)) \tag{7.16}$$

In the above, 7.15 provides a way of determing $I_\Delta g(x)$, and 7.16 provides a way to convert a functional equation (where all occurences of f are in the form $f(g(x))$ to a difference equation.

7.6. Table

Here, w is an arbitrary constant.

$$g(x) \qquad I_\Delta g(x) \tag{7.17}$$

$$x+h \qquad \frac{x}{h} \tag{7.18}$$

$$ax+b \qquad \log_a\left(\frac{x(a-1)}{b}+1\right) \tag{7.19}$$

$$hx \quad h>0 \qquad \log_h x \tag{7.20}$$

$$-hx \quad h>0 \qquad \frac{\ln x}{\pi i+\ln h} \tag{7.21}$$

$$x^h, \quad h>0 \qquad \log_h \log_w x \tag{7.22}$$

$$x^{-h}, \quad h>0 \qquad \frac{\ln \log_w x}{\pi i+\ln h} \tag{7.23}$$

$$h^x \qquad \operatorname{slog}_h(x) \tag{7.24}$$

$$h-x \qquad \frac{\ln(h-2x)}{\pi i} \tag{7.25}$$

$$\frac{h}{x}, \quad h\neq 0,1 \qquad \frac{\ln\left(\frac{2\ln x}{\ln h}-1\right)}{\pi i} \tag{7.26}$$

8. Trigonometric Equations

$$f(\cos(x)) + f(\sin(x)) = 1 \Rightarrow f(x) = \frac{2}{\pi}\cos^{-1} x + C \tag{8.1}$$

$$f(\cos(x)) + f(\sin(x)) = f(x) \Rightarrow f(x) = 2^{\frac{2}{\pi}\cos^{-1} x} C \tag{8.2}$$

Part II.

Transforms

Here, $\Delta(x)$ is the unit integer pulse function:

$$\Delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{8.3}$$

9. z-Transform

This section deals with the one-sided z-transform.

9.1. Properties

If $F(z) = \mathcal{Z}[f(x)]$ and $G(z) = \mathcal{Z}[g(x)]$, then

$$\mathcal{Z}[af(x) + bg(x)] = aF(z) + bG(z) \tag{9.1}$$
$$\mathcal{Z}[f(x-y)] = z^{-y}F(z) \tag{9.2}$$
$$\mathcal{Z}[f(-x)] = F\left(\frac{1}{z}\right) \tag{9.3}$$
$$\mathcal{Z}[a^x f(x)] = F\left(\frac{z}{a}\right) \tag{9.4}$$
$$\mathcal{Z}[\cos(ax)f(x)] = \frac{1}{2}\left[F\left(e^{ai}z\right) + F(e^{-ai}z)\right] \tag{9.5}$$
$$\mathcal{Z}[\sin(ax)f(x)] = \frac{i}{2}\left[F\left(e^{ai}z\right) - F(e^{-ai}z)\right] \tag{9.6}$$
$$\mathcal{Z}[x^k f(x)] = -z\frac{d}{dz}\mathcal{Z}[x^{k-1}f(x)] \tag{9.7}$$
$$\mathcal{Z}[\Delta f(x)] = (z-1)F(z) \tag{9.8}$$
$$\mathcal{Z}\left[\sum_{k=0}^{x} f(k)\right] = \frac{zF(z)}{z-1} \tag{9.9}$$
$$\mathcal{Z}\left[\sum_{k=0}^{x-1} f(k)\right] = \frac{F(z)}{z-1} \tag{9.10}$$
$$\mathcal{Z}[f(x) * g(x)] = F(z)G(z) \tag{9.11}$$
$$\mathcal{Z}[f(x)g(x)] = \frac{1}{2\pi i}\oint_C \frac{F(w)G(\frac{z}{w})}{z}dw \tag{9.12}$$
$$\mathcal{Z}[\mathcal{L}[f(x)]] = \int_0^\infty \frac{f(t)}{1-e^{-t}z^{-1}}dt \tag{9.13}$$

$$\lim_{z\to\infty} F(z) = f(0) \tag{9.14}$$
$$\lim_{x\to\infty} f(x) = \lim_{z\to 1}(1-z^{-1})F(z) \tag{9.15}$$
$$\sum_{x=0}^{\infty} f(x)g^*(x) = \frac{1}{2\pi i}\oint_C \frac{F(w)G(\frac{1}{w^*})}{z}dw \tag{9.16}$$

9.2. Pairs

9.2.1. Basics

$$\Delta(x) \to 1 \tag{9.17}$$

$$\Delta(x-n) \to z^{-n} \qquad z \neq 0 \tag{9.18}$$

$$1 \to \frac{z}{z-1} \tag{9.19}$$

$$u(x) \to \frac{z}{z-1} \tag{9.20}$$

$$u(x-k) \to \frac{z}{z^k(z-1)} \tag{9.21}$$

$$a^x \to \frac{z}{z-a} \tag{9.22}$$

$$\mathcal{F}(x) \to \frac{z}{z^2-z-1} \tag{9.23}$$

$$x \to \frac{z}{(z-1)^2} \tag{9.24}$$

$$x^2 \to \frac{z(z+1)}{(z-1)^3} \tag{9.25}$$

$$x^3 \to \frac{z(z^2+4z+1)}{(z-1)^4} \tag{9.26}$$

$$x^4 \to \frac{z(z^3+11z^2+11z+1)}{(z-1)^6} \tag{9.27}$$

$$x^5 \to \frac{z(z^4+26z^3+66z^2+26z+1)}{(z-1)^6} \tag{9.28}$$

$$x^{\langle n\rangle} \to \frac{n!z}{(z-1)^{n+1}} \tag{9.29}$$

$$xa^x \to \frac{az}{(z-a)^2} \qquad |z| > |a| \tag{9.30}$$

$$x^2a^x \to \frac{az(z+a)}{(z-a)^3} \qquad |z| > |a| \tag{9.31}$$

$$x^3a^x \to \frac{az(z^2+4az+a^2)}{(z-a)^4} \tag{9.32}$$

$$\frac{1}{x+1} \to \ln\left(\frac{z}{z-1}\right) \tag{9.33}$$

$$\frac{a^{x+1}}{x+1} \to \ln\left(\frac{z}{z-a}\right) \tag{9.34}$$

$$\frac{1-a^{x+1}}{x+1} \to \ln\left(\frac{z-a}{z-1}\right) \tag{9.35}$$

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^{\langle n\rangle} \to \sum_{n=0}^{\infty} \frac{a_n z}{(z-1)^{n+1}} \tag{9.36}$$

9.2.2. Trigonometric Functions

$$\sin(bx) \to \frac{z\sin b}{z^2 - 2z\cos b + 1} \tag{9.37}$$

$$\cos(bx) \to \frac{z(z - \cos b)}{z^2 - 2z\cos b + 1} \tag{9.38}$$

$$a^x \sin(bx) \to \frac{az\sin b}{z^2 - 2az\cos b + a^2} \qquad |z| > e^{-a} \tag{9.39}$$

$$a^x \cos(bx) \to \frac{z(z - a\cos b)}{z^2 - 2az\cos b + a^2} \qquad |z| > e^{-a} \tag{9.40}$$

$$\frac{\sin b(x+1)}{x+1} \to b + \operatorname{atan}\frac{\sin b}{z - \cos b} \tag{9.41}$$

$$\frac{\cos b(x+1)}{x+1} \to \ln\frac{z}{\sqrt{z^2 - 2z\cos b + 1}} \tag{9.42}$$

9.2.3. Hyperbolic Functions

$$\sinh(bx) \to \frac{z\sinh b}{z^2 - 2z\cosh b + 1} \qquad |z| > e^{-b} \tag{9.43}$$

$$\cosh(bx) \to \frac{z(z - \cosh b)}{z^2 - 2z\cosh b + 1} \qquad |z| > e^{-b} \tag{9.44}$$

$$a^x \sinh(bx) \to \frac{az\sinh b}{z^2 - 2az\cosh b + a^2} \tag{9.45}$$

$$a^x \cosh(bx) \to \frac{z(z - a\cosh b)}{1 - 2az\cosh b + a^2} \tag{9.46}$$

9.2.4. Discreet Trigonometric Functions

$$\operatorname{sind}(bx) \to \frac{\operatorname{sind}(b)z^{-1}}{1 - 2\operatorname{cosd}(b)z^{-1} + 2^b z^{-2}} \tag{9.47}$$

$$\operatorname{cosd}(bx) \to \frac{1 - \operatorname{cosd}(b)z^{-1}}{1 - 2\operatorname{cosd}(b)z^{-1} + 2^b z^{-2}} \tag{9.48}$$

$$\tag{9.49}$$

9.2.5. Gamma and Related Functions

$$\Gamma(x) \to^? e^{-z}C \tag{9.50}$$
$$\frac{a^x}{\Gamma(x+1)} \to e^{a/z} \tag{9.51}$$
$$\frac{a^{2x+1}}{\Gamma(2x+2)} \to \sinh\left(\frac{c}{z}\right) \tag{9.52}$$
$$\frac{a^{2x}}{\Gamma(2x+1)} \to \cosh\left(\frac{c}{z}\right) \tag{9.53}$$
$$\frac{(\ln a)^x}{\Gamma(x+1)} \to a^{1/z} \tag{9.54}$$
$$\binom{y}{x} a^{y-x} b^x \to \frac{(az+b)^y}{z^y} \tag{9.55}$$
$$\binom{x+y}{y} b^x \to \frac{z^{y+1}}{(z-b)^{y+1}} \tag{9.56}$$
$$G(x) \to^? \frac{e^{-z}C}{z-1} \tag{9.57}$$
$$xG(x) \to^? \frac{e^{-z}z^2C}{(z-1)^2} \tag{9.58}$$
$$\psi_0(x+1) - \psi_0(1) \to \frac{z}{z-1} \ln\left(\frac{z}{z-1}\right) \tag{9.59}$$
$$B(x,a) \to^? (z-1)^{a-1}C \tag{9.60}$$
$$\gamma(x,a) \to^? \mathrm{Ei}(a-z)e^z + Ce^z \tag{9.61}$$
$$\Gamma(x,a) \to^? -\mathrm{Ei}(a-z)e^z + Ce^z \tag{9.62}$$

9.2.6. Exponential Sums (?)

$$\sum \frac{\sin b(x+1)}{b(x+1)} \to \frac{1}{2b(z-1)} \ln\left(\frac{z-\cos b-\sin b}{z-\cos b+\sin b}\right) \tag{9.63}$$
$$\sum \frac{\cos b(x+1)}{b(x+1)} \to \frac{1}{2b(z-1)} \ln\left(z^2 - 2\cos bz + 2^b\right) \tag{9.64}$$
$$\mathrm{Bs}(b(x+1)) \to -\frac{\ln(1-2^b z^{-1})}{b(z-1)} \tag{9.65}$$
$$\mathrm{Ss}(b(x+1)) \to \frac{1}{2b(z-1)} \ln\left(\frac{z-\mathrm{cosd}(b)-\mathrm{sind}(b)}{z-\mathrm{cosd}(b)+\mathrm{sind}(b)}\right) \tag{9.66}$$
$$\mathrm{Cs}(b(x+1)) \to \frac{1}{2b(z-1)} \ln\left(z^2 - 2\,\mathrm{cosd}(b)z + 2^b\right) \tag{9.67}$$

9.2.7. Special Functions

The following functions are used in the table below:

- $J_k(t)$, Bessel polynomials
- $H_k(t)$, Hermite polynomials
- $L_k(t)$, Laguerre polynomials (?)
- $P_k(t)$, Legendre polynomials

- $T_k(t)$, Chebyshev polynomials

$$a^x P_x(t) \to \frac{z}{\sqrt{z^2 + -2taz + a^2}} \tag{9.68}$$

$$a^x P_x^m(t) \to \frac{(2m)! z^{m+1} (1-t^2)^{m/2} a^m}{2^m m! (z^2 - 2atz + a^2)^{m+\frac{1}{2}}} \tag{9.69}$$

$$\frac{P_x(t)}{x!} \to \exp\left(\frac{t}{z}\right) J_0\left(\frac{\sqrt{1-x^2}}{z}\right) \tag{9.70}$$

$$\frac{P_x^m(t)}{(x+m)!} \to (-1)^m \exp\left(\frac{t}{z}\right) J_m\left(\frac{\sqrt{1-x^2}}{z}\right) \tag{9.71}$$

$$a^x T_x(t) \to \frac{z(z-at)}{z^2 - 2atz + a^2} \tag{9.72}$$

$$\frac{L_x(t)}{x!} \to \frac{z e^{-t/(z-1)}}{z-1} \tag{9.73}$$

$$\frac{H_x(t)}{x!} \to e^{t/z - \frac{1}{2} z^2} \tag{9.74}$$

$$\frac{L_x^m}{x!} \to \frac{(-1)^m z}{(z-1)^{m+1}} \exp\left(\frac{-t}{z-1}\right) \tag{9.75}$$

$$\tag{9.76}$$

9.2.8. Sums

The following functions are used in the table below:

- $J_k(t)$, Bessel polynomials
- $H_k(t)$, Hermite polynomials
- $L_k(t)$, Laguerre polynomials
- $P_k(t)$, Legendre polynomials
- $T_k(t)$, Chebyshev polynomials

$$\sum_{k=0}^{x-1} \frac{1}{\Gamma(k+1)} \to \frac{e^{1/z}}{z-1} \tag{9.77}$$

$$\sum_{k=0}^{x} \frac{a^k b^{x-k}}{k!} \to \frac{e^{a/z} z}{z-b} \qquad b^2 < 1 \tag{9.78}$$

$$\sum_{k=0}^{x} a^k b^{x-k} J_k(t) \to \frac{z}{z-b} \exp\left(\frac{t(a^2+z^2)}{2az}\right) \qquad b^2 < 1; d \in \mathbb{R} \tag{9.79}$$

$$\sum_{k=0}^{x} \frac{a^k b^{x-k}}{k!} H_k(t) \to \frac{z}{z-b} \exp\left(-\frac{dz(2tz+a)}{2z^2}\right) \qquad b^2 < 1 \tag{9.80}$$

$$\sum_{k=0}^{x} \frac{a^k b^{x-k}}{k!} \frac{d}{dt} L_k(t) \to \frac{(-a)^m z^2}{(z-a)^{m+1}(z-b)} \exp\left(\frac{at}{1-z}\right) \qquad a^2 < 1; b^2 < 1 \tag{9.81}$$

$$\sum_{k=0}^{x} \frac{b^{x-k}\left[(-a)^k - (-c)^k\right]}{k} \to \frac{z}{z-b} \ln\left(\frac{z+c}{z+a}\right) \qquad a^2 < 1; b^2 < 1; c^2 < 1 \tag{9.82}$$

$$\sum_{k=0}^{x} a^k b^{x-k} \frac{d}{dt} P_k(t) \to \frac{(2m)! a^m}{2^m m!} \frac{z^{m+2}}{(z-b)(z^2 - 2atz + a^2)^{m+\frac{1}{2}}} \qquad a^2 < 1; b^2 < 1 \tag{9.83}$$

$$\sum_{k=0}^{x} a^k b^{x-k} T_k(t) \to \frac{z^2(z-at)}{(z-bz)(z^2-2atz+a)} \qquad a^2 < 1; b^2 < 1 \tag{9.84}$$

10. Binomial Transforms

10.1. Definition

The binomial transform $F(k) = \mathcal{B}[f(x)]$ of a function $f(x)$ is defined by:

$$F(k) = \sum_{x=0}^{k} (-1)^{k-x} \binom{k}{x} f(x) \tag{10.1}$$

The inverse transform $f(x) = \mathcal{B}^{-1}[F(x)]$ of a function $f(x)$ is defined by:

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} F(k) \tag{10.2}$$

10.2. Properties

Here, E is the shift operator: $E_x f(x) = f(x-1)$.

$$\mathcal{B}[af(x) + bg(x)] = a\mathcal{B}[f(x) + b\mathcal{B}[g(x)] \tag{10.3}$$

$$\mathcal{B}^{-1}[E_k f(k)] = f(-1) + xE_x\mathcal{B}^{-1}\left[\frac{f(k)}{k+1}\right] \tag{10.4}$$

$$\mathcal{B}^{-1}[kE_k f(k)] = xE_x\mathcal{B}^{-1}[f(k)] \tag{10.5}$$

$$\mathcal{B}^{-1}\left[\sum_{m=0}^{\infty} a_m k^{\langle m\rangle}\right] = 2^x \sum_{m=0}^{\infty} a_m 2^{-m} x^{\langle m\rangle} \tag{10.6}$$

$$f(x) = \sum_{k=0}^{\infty} F(k) \frac{x^{\langle k\rangle}}{k!} \tag{10.7}$$

$$F(k) = \Delta^k f(x)|_{x=0} \tag{10.8}$$

10.3. Pairs

10.3.1. Basics

$$1 \to \Delta(k) \tag{10.9}$$

$$x \to \Delta(k-1) \tag{10.10}$$

$$x^{\langle n\rangle} \to n!\Delta(k-n) \tag{10.11}$$

$$x! \to !k \tag{10.12}$$

$$2^x \to 1 \tag{10.13}$$

$$a^x \to (a-1)^k \tag{10.14}$$

$$\frac{1}{2^x} \to (-1)^k \frac{1}{2^k} \tag{10.15}$$

10.4. Pairs (Inverse)

10.4.1. Basics

$$1 \to 2^x \tag{10.16}$$

$$k \to x2^{x-1} \tag{10.17}$$

$$k^{\langle n \rangle} \to x^{\langle n \rangle} 2^{x-n} \qquad n \geq 0 \tag{10.18}$$

$$k^{\langle -1 \rangle} \to \frac{1}{x+1}(2^{x+1} - 1) \tag{10.19}$$

$$k^{\langle -2 \rangle} \to x^{\langle -2 \rangle}(2^{x+2} - x - 3) \tag{10.20}$$

$$k^{\langle -3 \rangle} \to x^{\langle -3 \rangle}(2^{x+3} - \frac{x^2 + 9x + 14}{2}) \tag{10.21}$$

$$k^{\langle -n \rangle} \to x^{\langle -2 \rangle} \left(2^{x+n} - \sum_{k=0}^{n-1} \binom{x+n}{k} \right) \tag{10.22}$$

$$a^k \to (a+1)^x \tag{10.23}$$

10.4.2. Discrete Trigonometric Functions

$$\operatorname{sind} k \to 2^x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} 2^{-2m-1} x^{\langle 2m+1 \rangle} \tag{10.24}$$

$$\operatorname{cosd} k \to 2^x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} 2^{-2m} x^{\langle 2m \rangle} \tag{10.25}$$

$$\tag{10.26}$$

10.4.3. Gamma and Related Functions

$$\psi_0(k+1) \to \gamma 2^x + 2^x \sum_{m=1}^{\infty} (-1)^m (m-1)! 2^{-m} x^{\langle m \rangle} \tag{10.27}$$

$$\Gamma(k+1) \to \sum_{k=0}^{x} x^{\langle k \rangle} \tag{10.28}$$

Bibliography

[CFD] Jordan, C. *Calculus of Finite Differences.* Chelsea Publishing Company, New York, 1947.

[HMF] Abramowitz, M. and Stegun, I.A. *Handbook of Mathematical Functions.* 1972.

[HM] Polyanin, A.D. and Manzhirov, A.V. *Handbook of Mathematics for Engineers and Scientists.* Chapman & Hall/CRC, USA, 2007.

[HMFI] A. Jeffrey, H. Dai. *Handbook of Mathematical Formulas and Integrals.* 4th Edition. Elsevier, 2008.

[HFTSP] Alexander D. Poularikas *The Handbook of Tables and Formulas for Signal Processing* CRC Press, 1999.

[NHMF] Olver, F.W.J, et al. *Nist Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[RAO] Roa, S. N. Additions to the z-Transform Table. *Electronics Letters.* September 1965 Vol. 1 No 7.